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Haar Function Based Estimates of the Star-Discrepancy of Plane Digital Nets

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Abstract. We apply the Haar function system to estimate the star-discrepancy of special digital (t, m, s)-nets in dimension $s = 2$. We use a basic technique based on discretization combined with an *exact* calculation of the discrete star-discrepancy.

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1. Introduction

In the present paper we show how to estimate the star-discrepancy of special digital (t, m, s)-nets using Haar functions. We apply the technique used by Hellekalek [5, 6, 7], who proved the inequality of Erdös-Turán-Koksma for Walsh- and Haar function systems.

A direct application of the latter inequality, which provides an upper bound for the discrepancy of a point set in the s-dimensional unit cube in terms of Weyl sums, yields, for arbitrary (t, m, s)-nets, no improvements of existing estimates of Niederreiter [16], which are due to weak upper bounds of the Weyl sums in general, see [2].

The power of our approach becomes apparent for special construction methods called digital nets. We applied the Haar function technique to estimate the star-discrepancy of special digital (t, m, s)-nets in dimension $s = 2$ which can be derived from the classical Hammersley point set by digit truncation. With this, we obtain a class of plane nets with steadily decreasing equidistribution property, from "optimal" $(0, m, 2)$-nets to the classical uniform lattice.

Our strategy provides a method for the exact calculation of the *discrete* star-discrepancy for such nets, and therefore yields best possible estimates and sometimes exact results of the star-discrepancy.

For further applications of Haar functions in (quasi) Monte Carlo methods and for related topics, see [2, 3, 8, 9, 12, 19].

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2. Basic Notations

Definition 2.1. Let $\mathscr{P} = (\mathbf{x}_n)_{n=0}^{N-1}$ be a point set in $[0,1[^s$. The star-discrepancy $D_N^*(\mathscr{P})$ of $\mathscr{P}$ is defined as

$$D_N^*(\mathscr{P}) := \sup_{J \in \mathscr{J}^*} \left| \frac{1}{N} \cdot \sharp\{n : \mathbf{x}_n \in J,\, 0 \leqslant n < N\} - \lambda_s(J) \right|, \tag{1}$$

where $\mathscr{J}^*$ denotes the class of all subintervals J of $[0,1[^s$ of the form $J = \prod_{i=1}^s [0, u_i[, 0 \leqslant u_i \leqslant 1, 1 \leqslant i \leqslant s, \sharp\mathscr{M}$ denotes the number of elements of a set $\mathscr{M}$ and λ_s the Lebesgue measure on $[0,1[^s$. Defining $f_J(\mathbf{x}) := \mathbf{1}_J(\mathbf{x}) - \lambda_s(J)$, $\mathbf{x} \in [0,1[^s$, with the characteristic function $\mathbf{1}_J$, yields

$$D_N^*(\mathscr{P}) = \sup_{J \in \mathscr{J}^*} |S_N(f_J, \mathscr{P})| \quad \text{where} \quad S_N(f_J, \mathscr{P}) := \frac{1}{N} \sum_{n=0}^{N-1} f_J(\mathbf{x}_n). \tag{2}$$

Consider integers M and a_i, $1 \leqslant i \leqslant s$, where $0 \leqslant a_i \leqslant M$ and $M > 0$. Using the smaller class $\mathscr{J}_M^*$ of all finite precision intervals $G = \prod_{i=1}^s [0, a_i/M[$, in (1) and (2), yields the concept of discrete star-discrepancy $E_{N,M}^*(\mathscr{P}) := \sup_{G \in \mathscr{J}_M^*} |S_N(f_G, \mathscr{P})|$, which provides an obvious equidistribution measure for suitable finite precision point sets.

Note that the *local discrepancy* $S_N(f_J, \mathscr{P})$ equals the Monte-Carlo approximation of $\int_{[0,1[^s} f_J \, d\lambda_s$. We will use the notation $S_N(g, \mathscr{P})$ for an arbitrary function g on $[0,1[^s$ as well. For orthogonal function systems $\mathscr{F} := \{\chi_{\mathbf{k}}\}$ the expression $S_N(\chi_{\mathbf{k}}, \mathscr{P})$ is called Weyl sum.

For the discrepancy estimates we use the Haar function system in base $b = 2$. The definitions and basic properties of Haar functions relative to an arbitrary integer base $b \geqslant 2$ are given in [6, 7]. We briefly recall the basic notations which will be used in the following sections.

For an integer $k \geqslant 0$ and an arbitrary number $x \in [0,1[$, let $k = \sum_{j=0}^{\infty} k_j b^j$ and $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, $k_j, x_j \in \{0, 1, \ldots, b-1\}$, $x_j \neq b-1$ for infintely many j, be the unique b-adic expansions of k and x in base b. For $g \in \mathbf{N}$ we define $k(g) := \sum_{j=0}^{g-1} k_j b^j$ and $x(g) := \sum_{j=0}^{g-1} x_j b^{-j-1}$. Further let $k(0) := 0$ and $x(0) := 0$. The support of a given Haar function $h_k, k \geqslant 0$, is equal to an elementary b-adic interval. We now define sets of integers k, for which such intervals have the same length (resolution).

Definition 2.2. (1) Let g be a nonnegative integer. Then $\Delta(g) := \{k \in \mathbf{N} : b^g \leqslant k < b^{g+1}\}$. Further, let $\Delta(-1) := \{0\}$, and the sets $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, and $\mathbf{N}_1 = \mathbf{N}_0 \cup \{-1\}$.

(2) If $\mathbf{g} = (g_1, \ldots, g_s)$, $s \geqslant 2$ and $g_i \in \mathbf{N}_1$, then $\Delta(\mathbf{g}) := \prod_{i=1}^s \Delta(g_i)$.

Definition 2.3. Let $e_b : \mathbf{Z}_b \to \mathbf{K}$, where $\mathbf{Z}_b = \{0, \ldots, b-1\}$ is the least residue system modulo b, and $\mathbf{K} := \{z \in \mathbf{C} : |z| = 1\}$, denote the function $e_b(a) := \exp(2\pi i \frac{a}{b})$, $(a \in \mathbf{Z}_b)$. The k-th Haar function h_k, $k \geqslant 0$, to the base b is defined as follows: If $k = 0$ $(g = -1)$, then $h_0(x) := 1\, \forall x \in [0,1[$. If $k \in \Delta(g)$, $g \geqslant 0$, then $h_k(x) := b^{\frac{g}{2}} \cdot e_b(a \cdot k_g)$ for $x \in [(b\,k(g) + a)/b^{g+1},$

$(b\,k(g)+a+1)/b^{g+1}[$ and $h_k(x) := 0$ otherwise. The k-th normalized Haar function H_k on $[0,1[$ is defined as $H_0 := h_0$ and, if $k \in \Delta(g), g \geqslant 0$, then $H_k := b^{-\frac{g}{2}} \cdot h_k$.

Hence, the support D_k of the k-th Haar function h_k is given as the following elementary b-adic interval : If $k = 0$, then $D_0 := [0,1[$. If $k \in \Delta(g)$, $g \geqslant 0$, then $D_k := [a/b^g, (a+1)/b^g[, a = k(g)$.

Definition 2.4. Let $\mathscr{H}_b := \{h_{\mathbf{k}} : \mathbf{k} := (k_1, \ldots, k_s) \in \mathbf{N}_0^s\}$ denote the Haar function system to the base b on the s-dimensional torus $[0,1[^s$, $s \geqslant 1$. The $\mathbf{k}$-th Haar function $h_{\mathbf{k}}$ is defined as $h_{\mathbf{k}}(\mathbf{x}) := \prod_{i=1}^{s} h_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1[^s$. The normalized version $H_{\mathbf{k}}$ is defined in the same way and the supports of $h_{\mathbf{k}}$ and $H_{\mathbf{k}}$ are given by the elementary b-adic intervals $D_{\mathbf{k}} := \prod_{i=1}^{s} D_{k_i}$.

Sobol' [19] introduced the concept of (t,m,s)-nets in base 2, closely coherent with an application of classical Haar functions in the theory of quasi-Monte Carlo methods. The main contributions concerning development and analysis of such nets are due to Niederreiter. He has introduced arbitrary bases, efficient construction methods for such point sets and a comprehensive theory [13, 14, 16, 18].

Definition 2.5. Let $0 \leqslant t \leqslant m$ and $b \geqslant 2$ be integers. A (t,m,s)-net in base b is a point set $\mathscr{P}$ consisting of b^m points in $[0,1[^s$ such that every elementary interval I in base b with volume $\lambda_s(I) = b^{t-m}$ contains exactly b^t points of $\mathscr{P}$.

Digital nets provide efficient construction methods of (t,m,s)-nets. They are defined in the following way.

Definition 2.6. Let $b \geqslant 2$ be a given base. For $1 \leqslant i \leqslant s$, let $C^{(i)}$ be $m \times m$ matrices over $\mathbf{Z}_b$. In the following every integer n with $0 \leqslant n < b^m$ and digit expansion $\sum_{i=0}^{m-1} n_i b^i$, $n_i \in \mathbf{Z}_b$, is identified with the vector $\vec{n} = (n_0, \ldots, n_{m-1})^t \in \mathbf{Z}_b^m$, and each $x \in [0,1[$ with finite digit expansion $x = \sum_{i=0}^{m-1} x_i/b^{i+1}$, $x_i \in \mathbf{Z}_b$, is identified with $\vec{x} = (x_0, \ldots, x_{m-1})^t \in \mathbf{Z}_b^m$. Consider $\vec{x}_n^{(i)} = C^{(i)} \cdot \vec{n}$ for $0 \leqslant n < b^m$ and $1 \leqslant i \leqslant s$. Then we obtain the following point set $\mathscr{P} \in [0,1[^s$

$$\mathscr{P} = \{\mathbf{x}_n : \mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)}), \quad 0 \leqslant n < b^m\}. \tag{3}$$

These point sets were defined in [10, 11] (in a more general form). The general construction principle was introduced by Niederreiter (see [14, 15, 16, 18]).

Conditions [14, Section 6] were given for $\mathscr{P}$ to be a (t,m,s)-net in base b. For example, if b is prime and $c_1^{(i)}, \ldots, c_m^{(i)}$ are the row vectors of $C^{(i)}$, then $\mathscr{P}$ is a (t,m,s)-net in base b if and only if for all $g_1, \ldots, g_s \in \mathbf{N}_0$ with $g_1 + \cdots + g_s = m - t$, the set of vectors $\{c_j^{(i)} : 1 \leqslant j \leqslant g_i,\ 1 \leqslant i \leqslant s\}$ is assumed to be linearly independent over $\mathbf{Z}_b$. Concrete examples of digital (t,m,s)-nets can be found in [9, 11, 14].

3. Examples of Plane Digital Nets

The conditions above yield the following simple $(t,m,2)$-nets $\mathscr{P}_t$, defined by the matrices

$$C^{(1)} := \begin{bmatrix} 000\cdots000\cdots00\mathbf{1} \\ 000\cdots000\cdots0\mathbf{1}0 \\ 000\cdots000\cdots\mathbf{1}00 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ 000\cdots00\mathbf{1}\cdots000 \\ 000\cdots0\mathbf{1}0\cdots000 \\ \left.\begin{matrix} 000\cdots000\cdots000 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ 000\cdots000\cdots000 \end{matrix}\right\} t \end{bmatrix} \qquad C^{(2)} := \begin{bmatrix} \mathbf{1}00\cdots000\cdots000 \\ 0\mathbf{1}0\cdots000\cdots000 \\ 00\mathbf{1}\cdots000\cdots000 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ 000\cdots\mathbf{1}00\cdots000 \\ 000\cdots0\mathbf{1}0\cdots000 \\ \left.\begin{matrix} 000\cdots000\cdots000 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ 000\cdots000\cdots000 \end{matrix}\right\} t \end{bmatrix}$$

The net $\mathscr{P}_0 := \{\mathbf{x}_n = (0.n_{m-1}\dots n_0, 0.n_0\dots n_{m-1}) : 0 \leqslant n < b^m\}$ is called the *Hammersley point set* in base b which is well-known in the theory of uniform distribution of sequences modulo one [16]. The calculation of the exact discrepancy of $\mathscr{P}_0$ in base $b = 2$ is carried out in [4], and for arbitrary bases in [1]. If we truncate t bits of each coordinate of the Hammersley point set, we obtain $\mathscr{P}_t$, $t \geqslant 1$. Note that for even m and $t = m/2$ the uniform lattice with 2^m points in $[0,1[^2$ is obtained. Hence $\mathscr{P}_t$, for increasing $t, 0 \leqslant t \leqslant m/2$, provide examples of point sets with steadily decreasing equidistribution property, from "optimal" $(0,m,2)$-nets to the classical uniform lattice.

In the following let $b = 2, m > 2, 0 \leqslant t \leqslant \lfloor m/2 \rfloor$ and $M = 2^\gamma, \gamma \in \mathbf{N}$. Larger values of t lead to a duplication of points. In other words, different values of n lead to the same point.

3.1. Discrepancy Estimation of $\mathscr{P}_t$. We follow a basic technique [7, 16] to estimate the discrepancy of $\mathscr{P}_t$ and, as a first step, approximate $J \in \mathscr{J}$ by an inner interval G_1 and an outer interval G_2 in $[0,1[^s$ where these intervals have the form $G := \prod_{i=1}^s [0, a_i/M[, 0 \leqslant a_i \leqslant M$. The approximation may be realized in the following way: Let $J := \prod_{i=1}^s [0, u_i[, G_1 := \prod_{i=1}^s [0, v_i[$ and $G_2 := \prod_{i=1}^s [0, w_i[$. If $u_i = a_i/M, 0 \leqslant a_i \leqslant M$, we set $v_i = w_i = u_i$, otherwise let $v_i := u_i(\gamma)$ and $w_i := u_i(\gamma) + 1/M$. Hence we get

$$|S_N(f_J, \mathscr{P}_t)| \leqslant (\lambda_s(G_2) - \lambda_s(G_1)) + \max\{|S_N(f_{G_1}, \mathscr{P}_t)|, |S_N(f_{G_2}, \mathscr{P}_t)|\}. \tag{4}$$

Using [16, Lemma 3.9] to estimate the *discretization error* $(\lambda_s(G_2) - \lambda_s(G_1))$ yields

$$D_N^*(\mathscr{P}_t) \leqslant 1 - \left(1 - \frac{1}{M}\right)^s + E_{N,M}^*(\mathscr{P}_t). \tag{5}$$

From [7, Lemma 3.3] it follows that the Haar series of the function f_G is finite and, therefore,

$$S_N(f_G, \mathscr{P}_t) = \sum_{\mathbf{k} \in \Delta_\gamma^*} \widehat{\mathbf{1}_G}(\mathbf{k}) \cdot S_N(h_\mathbf{k}, \mathscr{P}_t), \tag{6}$$

with Haar coefficients $\widehat{\mathbf{1}_G}(\mathbf{k}) := \int_{[0,1[^s} \mathbf{1}_G \cdot h_\mathbf{k}\, d\lambda_s$ and $\Delta_\gamma^* := \{\mathbf{k} = (k_1, \dots, k_s) \in \mathbf{Z}^s : 0 \leqslant k_i < M,\ 0 \leqslant i \leqslant s\} \setminus \{\mathbf{0}\}$.

The definition of (t,m,s)-nets easily yields $S_N(H_\mathbf{k}, \mathscr{P}_t) = 0$ for all $\mathbf{k} \in \Delta(\mathbf{g})$, $\mathbf{g} \in \mathbf{N}_1^2 \setminus (-1,-1)$ with $\sum_{i=1}^2 (g_i + 1) \leqslant m - t$. The latter property and the fact that all points of $\mathscr{P}_t$ have common denominator 2^{m-t} motivate a discretization with $M := 2^\gamma$, $\gamma = m - t$.

Now, let $G := [0, \alpha[\times[0, \beta[$ with $\alpha = a/M$ and $\beta = b/M, 1 \leqslant a, b < M$. We partition sum (6) into parts with different resolution. From the Weyl sum property above it follows that we only have to consider resolution vectors $\mathbf{g} = (g_1, g_2)$ with $0 \leqslant g_1, g_2 < \gamma$ and $\gamma - 1 \leqslant g_1 + g_2 \leqslant 2(\gamma - 1)$. Hence for the calculation of the discrete discrepancy $E^*_{N,M}(\mathscr{P}_t), N = 2^m$, we must analyze

$$S_N(f_G, \mathscr{P}_t) = \sum_{\substack{g_1, g_2 = 0 \\ g_1 + g_2 \geqslant \gamma - 1}}^{\gamma - 1} 2^{(g_1+g_2)/2} \sum_{\mathbf{k} \in \Delta(\mathbf{g})} \widehat{\mathbf{1}_G}(\mathbf{k}) \cdot S_N(H_{\mathbf{k}}, \mathscr{P}_t). \tag{7}$$

3.2. Haar Coefficients and Weyl Sums. Let $\mathbf{k} := (k, l) \in \Delta(\mathbf{g})$. Hellekalek [6, Lemma 1] provides the following information on the Haar coefficients $\widehat{\mathbf{1}_G}(\mathbf{k}) := \widehat{\mathbf{1}_{[0,\alpha[}}(k) \cdot \widehat{\mathbf{1}_{[0,\beta[}}(l)$. Let $k \in \Delta(g)$. For $I := [0, \delta[, \delta := 0.\delta_0\delta_1 \ldots \delta_{m-t-1}$ and $d(\delta, g) := (\delta - \delta(g + 1))$ it follows that

$$\widehat{\mathbf{1}_I}(k) = \begin{cases} 2^{\frac{g}{2}} d(\delta, g) & : \quad k(g) = \delta_{g-1} + \delta_{g-2}2 + \cdots + \delta_0 2^{g-1}, \\ & \quad\;\; \delta_g = 0 \\ 2^{\frac{g}{2}}(\frac{1}{2^{g+1}} - d(\delta, g)) & : \quad k(g) = \delta_{g-1} + \delta_{g-2}2 + \cdots + \delta_0 2^{g-1}, \\ & \quad\;\; \delta_g = 1 \\ 0 & : \quad \text{otherwise} \end{cases} \tag{8}$$

For the calculation of the Weyl sums we distinguish between two cases:

(a) Let $g_1 + g_2 = m - t + j$, $j \in \{-1, 0, \ldots, t - 1\}$. This is attained by $g_1 \in \{j + 1, \ldots, m - t - 1\}$ and $g_2 = m - t + j - g_1$. A point $\mathbf{x}_n := (x, y)$ is contained in the support $D_{\mathbf{k}}$ for $\mathbf{k} := (k, l) \in \Delta(\mathbf{g})$, if and only if

$$n_{m-1} = k_{g_1-1}, \ldots, n_{m-g_1} = k_0, \quad n_0 = l_{g_2-1}, \ldots, n_{m-g_1+j-t-1} = l_0,$$

and the $n_{m-t+j-g_1}, \ldots, n_{m-g_1-1}$ are arbitrary within $\vec{n}$. Thus $D_{\mathbf{k}}$ contains exactly 2^{t-j} points $\mathbf{x}_n$ with $x_{g_1} = n_{m-g_1-1}$ and $y_{g_2} = n_{m-g_1+j-t}$.

If $j = t - 1$ then $x_{g_1} = y_{g_2}$ and therefore

$$S_N(H_{\mathbf{k}}, \mathscr{P}_t) = 1/2^{m-1} \quad \text{for all} \quad \mathbf{k} \in \Delta(\mathbf{g}). \tag{9}$$

Let $j \leqslant t - 2$. Because of $m - g_1 - 1 \neq m - g_1 - t + j$ we obtain $S_N(H_{\mathbf{k}}, \mathscr{P}_t) = 0$ for all $\mathbf{k} \in \Delta(\mathbf{g})$.

(b) Let $g_1 + g_2 = m + j$, $0 \leqslant j \leqslant m - 2t - 1$, more precisely $g_1 \in \{t + j + 1, \ldots, m - t - 1\}$ and $g_2 = m + j - g_1$. Similarly as in case (a) we have $\mathbf{x}_n := (x, y) \in D_{\mathbf{k}}$, $\mathbf{k} \in \Delta(\mathbf{g})$, if and only if (here we already use the relevant vectors $\mathbf{k}$ obtained from property (8))

$$n_{m-1} = \alpha_0, \ldots, n_{m-g_1} = \alpha_{g_1-1}, \quad n_0 = \beta_0, \ldots, n_{m-g_1+j-1} = \beta_{g_2-1},$$

but now, all digits of $\vec{n}$ are fixed. If $\alpha_{g_1-1} \neq \beta_{g_2-j}$ or $\alpha_{g_1-2} \neq \beta_{g_2-j+1}$ or $\ldots$ or $\alpha_{g_1-j} \neq \beta_{g_2-1}$ then $D_{\mathbf{k}}$ is empty, otherwise $D_{\mathbf{k}}$ contains exactly one point $\mathbf{x}_n \in \mathscr{P}_t$ with $x_{g_1} = n_{m-g_1-1} = \beta_{g_2-j-1}$ and $y_{g_2} = n_{m-g_1+j} = \alpha_{g_1-j-1}$. Hence for $j = 0$ we obtain

$$S_N(H_{\mathbf{k}}, \mathscr{P}_t) = \begin{cases} 2^{-m} & : \quad \alpha_{g_1-1} = \beta_{g_2-1} \\ -2^{-m} & : \quad \alpha_{g_1-1} \neq \beta_{g_2-1} \\ 0 & : \quad \text{otherwise,} \end{cases} \tag{10}$$

and for $j \geqslant 1$,

$$S_N(H_{\mathbf{k}}, \mathscr{P}_t) = \begin{cases} 2^{-m} & : \alpha_{g_1-1} = \beta_{g_2-j}, \ldots, \alpha_{g_1-j} = \beta_{g_2-1}, \\ & \quad \alpha_{g_1-j-1} = \beta_{g_2-j-1} \\ -2^{-m} & : \alpha_{g_1-1} = \beta_{g_2-j}, \ldots, \alpha_{g_1-j} = \beta_{g_2-1}, \\ & \quad \alpha_{g_1-j-1} \neq \beta_{g_2-j-1} \\ 0 & : \text{otherwise.} \end{cases} \tag{11}$$

3.3. The Discrete Discrepancy of $\mathscr{P}_t$. Motivated by the cases above, we have to concentrate our consideration to

$$S_N(f_G, \mathscr{P}_t) = \sum_{\substack{g_1, g_2 = t \\ g_1 + g_2 = m-1}}^{m-t-1} 2^{(m-1)/2} \sum_{\mathbf{k} \in \Delta(\mathbf{g})} \widehat{\mathbf{1}_G}(\mathbf{k}) \cdot S_N(H_{\mathbf{k}}, \mathscr{P}_t) \tag{12}$$

$$+ \sum_{\substack{g_1, g_2 = t+1 \\ g_1 + g_2 = m}}^{m-t-1} 2^{m/2} \sum_{\mathbf{k} \in \Delta(\mathbf{g})} \widehat{\mathbf{1}_G}(\mathbf{k}) \cdot S_N(H_{\mathbf{k}}, \mathscr{P}_t) \tag{13}$$

$$+ \sum_{j=1}^{m-2t-2} \sum_{\substack{g_1, g_2 = t+j+1 \\ g_1 + g_2 = m+j}}^{m-t-1} 2^{(m+j)/2} \sum_{\mathbf{k} \in \Delta(\mathbf{g})} \widehat{\mathbf{1}_G}(\mathbf{k}) \cdot S_N(H_{\mathbf{k}}, \mathscr{P}_t). \tag{14}$$

Halton and Zaremba [4] derived the exact discrepancy for $\mathscr{P}_0$. The intervals $J_1 := [0, \alpha^*[\times[0, \beta^*[$ and $J_2 := [0, \alpha^{**}[\times[0, \beta^{**}[$ where the supremum in (1) is attained, are given by

$$\alpha^* = \frac{2^m - (-1)^m}{3 \cdot 2^{m-1}} \quad \text{and} \quad \alpha^{**} = \frac{5 \cdot 2^{m-2} + (-1)^m}{3 \cdot 2^{m-1}}, \tag{15}$$

where β^* and β^{**} are equal to α^* and α^{**}, respectively in that order when m is even, and in the reversed order when m is odd. These intervals obviously will play a central role in our strategy below.

The discrepancy estimation of $\mathscr{P}_t$ needs to distinguish between several cases (similar as for $\mathscr{P}_0$ in [1, 4]). We will give a detailed derivation only for even m, where additionally, $m - 2t - 6$ equals a multiple of four (for the reason of the latter restriction, see below). The remaining cases can be done in almost the same way.

Since we know the "maximal" intervals for the discrepancy of $\mathscr{P}_0$, we can derive the corresponding intervals for the *discrete discrepancy* of $\mathscr{P}_0$. We slightly change the discretization step (4) for $\mathscr{P}_0$ and use the inner interval $G_1 := J_1$ and outer interval $G_2 := [0, \alpha'[\times[0, \beta'[$ with $\alpha' = \beta' = \alpha^* + 1/2^m = (2 \cdot 2^m + 1)/(3 \cdot 2^m) = 0.10101\ldots0101011$. It follows that the *discrete* discrepancy for $\mathscr{P}_0$ equals $|S_N(f_{G_2}, \mathscr{P}_t)|$.

The magnitude of the sums in (12,13,14) depend only on the Haar coefficients (8) and therefore on the binary expansion of α' and β' (the Weyl sums are constant for each of the single sums, compare the cases I–III for $t = 0$ below).

Hence, if we change to $\mathscr{P}_t$, $t \geqslant 1$, the expression (12,13,14) will attain the maximum for α and β with the following sequences of digits

$$(\alpha_t, \alpha_{t+1}, \ldots, \alpha_{m-t-2}, \alpha_{m-t-1}) = (1, 0, 1, 0, 1, \ldots, 0, 1, 0, 1, 1)$$
$$(\beta_t, \beta_{t+1}, \ldots, \beta_{m-t-2}, \beta_{m-t-1}) = (1, 0, 1, 0, 1, \ldots, 0, 1, 0, 1, 1).$$

The remaining digits $\alpha_0, \ldots, \alpha_{t-1}, \beta_0, \ldots, \beta_{t-1}$, can be chosen arbitrarily. Therefore the maximum intervals for the discrete star-discrepancy for P_t in general are $G_2 := [0, \alpha[\times[0, \beta[$ with

$$\alpha = \frac{\phi}{2^t} + \frac{2 \cdot 2^{m-2t} + 1}{3 \cdot 2^{m-t}}, \quad \beta = \frac{\phi'}{2^t} + \frac{2 \cdot 2^{m-2t} + 1}{3 \cdot 2^{m-t}}, \quad 0 \leqslant \phi, \phi' < 2^t. \tag{16}$$

Note that the marginal points of corresponding inner intervals G_1 equal $\alpha - 1/2^{m-t}$ and $\beta - 1/2^{m-t}$.

From these intervals (i.e. from the binary sequences above) we can derive the exact values of the Haar coefficients. Suggested by (12,13,14) we distinguish between three cases:

Case I: Let $G = G_2$ and $\mathbf{k} \in \Delta(\mathbf{g})$ with $g_1 + g_2 = m - 1$ and $g_1 = t + 2i + 1$, $0 \leqslant 2i \leqslant (m - 2t - 6)/2$. From (8) we know that there is only one $\mathbf{k} \in \Delta(\mathbf{g})$ with $\widehat{\mathbf{1}_G}(\mathbf{k}) \neq 0$, and such a $\mathbf{k}$ depends on the binary representation of α and β. From the binary sequences above it follows that

$$\widehat{\mathbf{1}_G}(\mathbf{k}) = 2^{(m-1)/2} \cdot \left(\frac{1}{2^{t+2i+3}} + \frac{1}{2^{t+2i+5}} + \cdots + \frac{1}{2^{m-t-1}} + \frac{1}{2^{m-t}} \right) \cdot \\ \cdot \left[\frac{1}{2^{m+t-2i-1}} - \left(\frac{1}{2^{m-t-2i+1}} + \frac{1}{2^{m-t-2i+3}} + \cdots + \frac{1}{2^{m-t-1}} + \frac{1}{2^{m-t}} \right) \right]$$

and therefore

$$\widehat{\mathbf{1}_G}(\mathbf{k}) = 2^{(m-1)/2} \cdot \frac{1}{9} \cdot \frac{1}{2^{2m-2t}} \cdot (2^{m-2t-2i-1} + 1) \cdot (2^{2i+2} - 1). \tag{17}$$

The same strategy for $g_1 = t + 2i + 2, 0 \leqslant 2i \leqslant (m - 2t - 6)/2$, yields

$$\widehat{\mathbf{1}_G}(\mathbf{k}) = 2^{(m-1)/2} \cdot \frac{1}{9} \cdot \frac{1}{2^{2m-2t}} \cdot (2^{m-2t-2i-2} - 1) \cdot (2^{2i+3} + 1), \tag{18}$$

for only one $\mathbf{k} \in \Delta(\mathbf{g})$. Furthermore we have $S_N(H_{\mathbf{k}}, \mathscr{P}_t) = 1/2^{m-1}$ for all $\mathbf{k} \in \Delta(\mathbf{g})$ and therefore

$$(12) = \frac{2}{9} \frac{1}{2^{2(m-t)}} \sum_{i=0}^{(m-2t-6)/4} \left(\frac{2^{m-2t}}{2^{2i+1}} + 1 \right) (2^{2i+2} - 1) + \left(\frac{2^{m-2t}}{2^{2i+2}} - 1 \right) (2^{2i+3} + 1) \\ + \frac{2(2^{m-2t} - 1)}{3 \cdot 2^{2(m-t)}} \\ = \frac{2}{9} \cdot \frac{m - 2t}{2^m} - \frac{1}{9} \cdot \frac{m - 2t}{2^{2(m-t)}} + \frac{4}{27} \cdot \frac{1}{2^m} - \frac{4}{27} \cdot \frac{1}{2^{2(m-t)}}.$$

Case II: Let $\mathbf{k} \in \Delta(\mathbf{g})$ with $g_1 + g_2 = m$. If we proceed in a similar way as in the upper case and distinguish between $g_1 = t + 2i + 2$ and $g_1 = t + 2i + 3$, $0 \leqslant 2i \leqslant (m-2t-6)/2$, then, with $S_N(H_{\mathbf{k}}, \mathscr{P}_t) = 1/2^m$, for the relevant $\mathbf{k}$, we obtain

$$(13) = \frac{2}{9} \frac{1}{2^{2(m-t)}} \sum_{i=0}^{(m-2t-6)/4} \left(\frac{2^{m-2t}}{2^{2i+2}} - 1 \right) (2^{2i+2} - 1) + \left(\frac{2^{m-2t}}{2^{2i+3}} + 1 \right) (2^{2i+3} + 1) \\ + \frac{2(2^{m-2t-1} + 1)}{3 \cdot 2^{2(m-t)}} - \frac{(2^{(m-2t)/2} + 1)^2}{9 \cdot 2^{2(m-t)}} \\ = \frac{1}{9} \cdot \frac{m - 2t}{2^m} + \frac{1}{9} \cdot \frac{m - 2t}{2^{2(m-t)}} - \frac{1}{27} \cdot \frac{1}{2^m} + \frac{1}{27} \cdot \frac{1}{2^{2(m-t)}}.$$

Case III: Let $\mathbf{k} \in \Delta(\mathbf{g})$ with $g_1 + g_2 = m + j$, $1 \leqslant j \leqslant m - 2t - 2$. From (11) we deduce that $S_N(H_{\mathbf{k}}, \mathscr{P}_t) = 0$ for all j and therefore $(14) = 0$.

The cases I, II and III yield the value for discrete discrepancy for even m:

$$E^*_{N,M}(\mathscr{P}_t) = \frac{1}{3} \cdot \frac{m - 2t}{2^m} + \frac{1}{9} \cdot \frac{1}{2^m} - \frac{1}{9} \cdot \frac{1}{2^{2(m-t)}} \tag{19}$$

3.4. Final Result and Conclusions. If we carry out almost the same calculations for the remaining cases (m odd, and some other cases for even m) we obtain the following final result:

Proposition 3.1. *For $\mathscr{P}_t$, $0 \leqslant t \leqslant \lfloor m/2 \rfloor$, the digital $(t, m, 2)$-nets in base $b = 2$, defined at the beginning of Section 3, we get $(N = 2^m,\ M = 2^{m-t})$:*

$$E^*_{N,M}(\mathscr{P}_t) = \frac{1}{3} \cdot \frac{m - 2t}{2^m} + \frac{1}{9} \cdot \frac{1}{2^m} - \frac{(-1)^m}{9} \cdot \frac{1}{2^{2(m-t)}},$$

and therefore

$$D^*_N(\mathscr{P}_t) \leqslant 1 - \left(1 - \frac{1}{2^{m-t}}\right)^2 + E^*_{N,M}(\mathscr{P}_t).$$

Remark 3.1. (i) Let again m be even. If we use the intervals (16) where the maximum in (4) occurs and calculate the discretization error $(\lambda_s(G_2) - \lambda_s(G_1)) = (3 \cdot 2^{m+t+1} - 2^{m+1} - 2^{2 \cdot t})/(3 \cdot 2^{2m})$, then we obtain

$$D^*_N(\mathscr{P}_t) \geqslant \frac{1}{3} \cdot \frac{m - 2t}{2^m} - \frac{5}{9} \cdot \frac{1}{2^m} + \frac{2}{2^{m-t}} - \frac{4}{9} \cdot \frac{1}{2^{2(m-t)}}. \tag{20}$$

(ii) From [4] we know that for $t = 0$ the latter expression equals the exact discrepancy of $\mathscr{P}_0$. Further, from [16, Thm. 3.14], we deduce that $D^*_N(\mathscr{P}_{m/2}) = 1 - (1 - \frac{1}{2^{m/2}})^2$ which equals expression (20) for $t = m/2$. Hence our estimates of the star-discrepancy are best possible.

(iii) With increasing quality parameter t, a remarkable interaction between the discretization error above and the discrete star-discrepancy occurs. Whereas for $t = 0$ both values have almost the same magnitude, for increasing parameter t the value of the discrete star-discrepancy vanishes and the discretization error increases until it attains the exact value of the star-discrepancy.

(iv) Consider the point set $\mathscr{P}_0$. There are two different intervals where the supremum in (1) occurs. In our case we obviously get the same number of "maximal" intervals defined by α' and $\alpha'' = 1 - \alpha'$. For $\mathscr{P}_t$, $0 \leqslant t \leqslant m/2$ there are 2^{t+1} such intervals.

4. Summary and Outlook

Our examples show that Haar functions can be applied effectively to calculate the discrete star-discrepancy of simple digital (t, m, s)-nets. The corresponding arithmetic is elementary, but rather lengthy, and similarly to former discrepancy calculations of related point sets [1, 4], needs to distinguish between several cases.

The previous knowledge of the intervals, where the discrepancy of $\mathscr{P}_0$ is attained [4], provides additional simplification in our case. Without this information one has to find the maximum of expression (7), which is an additional effort. Nevertheless, the Haar representation of local discrepancy (7) of such nets enables additional information for a possible classification of related digital $(t,m,2)$-nets with respect to discrepancy.

As an example, consider a different point set $\mathscr{P}'_t$, defined with the same matrix $C^{(1)}$ and a slightly modified $C^{(2)}$, where only the first column is changed to $(1,1,\ldots,1,1)^T$. For $\mathscr{P}'_t$, if we proceed in the same way as in Subsection 3.2, slightly different Weyl sums are obtained. Consider the same cases as in Subsection 3.2, then in case **(a)**, for $j=t-1$, we get

$$S_N(H_{\mathbf{k}},\mathscr{P}'_t)=\begin{cases} 2^{-(m-1)} & : \quad \beta_0=0 \\ -2^{-(m-1)} & : \quad \beta_0=1, \end{cases} \tag{21}$$

and $S_N(H_{\mathbf{k}},\mathscr{P}'_t)=0$ for $j\leqslant t-2$. In case (**b**) and $j=0$, we get

$$S_N(H_{\mathbf{k}},\mathscr{P}'_t)=\begin{cases} 2^{-m} & : \quad \alpha_{g_1-1}=\beta_{g_2-1}-\beta_0 \\ -2^{-m} & : \quad \alpha_{g_1-1}\neq\beta_{g_2-1}-\beta_0 \\ 0 & : \quad \text{otherwise}, \end{cases}$$

and for $j\geqslant 1$,

$$S_N(H_{\mathbf{k}},\mathscr{P}'_t)=\begin{cases} 2^{-m} & : \quad \alpha_{g_1-1}=\beta_{g_2-j}-\beta_0,\ldots,\alpha_{g_1-j}=\beta_{g_2-1}-\beta_0, \\ & \quad\ \ \alpha_{g_1-j-1}=\beta_{g_2-j-1}-\beta_0 \\ -2^{-m} & : \quad \alpha_{g_1-1}=\beta_{g_2-j}-\beta_0,\ldots,\alpha_{g_1-j}=\beta_{g_2-1}-\beta_0, \\ & \quad\ \ \alpha_{g_1-j-1}\neq\beta_{g_2-j-1}-\beta_0 \\ 0 & : \quad \text{otherwise}. \end{cases}$$

The change of matrix $C^{(2)}$ obviously reflects in the variance of the Weyl sums (changing the first column of $C^{(2)}$ implies an additional dependence of the first bit of β in $S_N(H_{\mathbf{k}},\mathscr{P}'_t)$).

For the ease of further explanation let m be even and $t=0$. Using $G=[0,\alpha''[^2$, $\alpha''=1-\alpha'=0.01010\ldots0101$, in the calculation of $S_N(f_G,\mathscr{P}'_0)$, in exactly the same way as in Subsection 3.3, yields

$$E^*_{N,M}(\mathscr{P}'_0)\geqslant\frac{1}{3}\cdot\frac{m}{2^m}+\frac{1}{9}\cdot\frac{1}{2^m}-\frac{1}{9}\cdot\frac{1}{2^{2m}}, \tag{22}$$

For $G=[0,\alpha'[^2$, $\alpha'=0.1010\ldots101011$, a smaller value of local discrepancy $S_N(f_G,\mathscr{P}'_0)$ is obtained, since the first bit of α' equals $\alpha'_0=1$.

Expression (22) shows that the change of matrix $C^{(2)}$ was not powerful enough to get an improvement of discrepancy. The Weyl sums for $\mathscr{P}'_t$ remained almost the same as for $\mathscr{P}_t$ except of an remarkable variance in case (a) (compare equations (9) and (21)).

Consider the class of $(0,m,2)$-nets $\mathscr{P}''_0$ with the same matrix $C^{(1)}$ as for $\mathscr{P}_0$ and matrix $C^{(2)}$ where the lower left triangle of $C^{(2)}$ is arbitrarily arranged with zeros and ones. For such nets $\mathscr{P}''_0$, the Weyl sums behave almost the same as for $\mathscr{P}'_0$ above, only the expressions $\beta_l-\beta_0$ change to somewhat larger expressions dependent on the digits of β and the entries of matrix $C^{(2)}$.

A closer look at (12,13,14) indicates that $\mathscr{P}_0$, and also $\mathscr{P}'_0$, attain the largest possible (discrete) discrepancy of all point sets $\mathscr{P}''_0$, since the Weyl sums achieve their maximal positive values, constant in each single sum in (12,13,14), and the Haar coefficients are also positive and globally maximal for the above intervals G.

Finally we want to ask: Is there a net $\mathscr{P}_0''$ with lower discrepancy as $\mathscr{P}_0$, and if (presumably) yes, which nets $\mathscr{P}_0''$ have smallest discrepancy? In other words: How dense has the matrix $C^{(2)}$ to be filled up with ones to get a net with optimal distribution. Note that an arbitrary digital $(0, m, 2)$-net can always be expressed with matrix $C^{(1)}$ and a transformed matrix $C^{(2)}$.

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