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QUASI-MONTE CARLO METHODS FOR NUMERICAL INTEGRATION OF MULTIVARIATE HAAR SERIES II *

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Abstract.

The present paper contains a comparison of different classes of multivariate Haar series that have been studied with respect to numerical integration, new properties of E_s^α-classes and numerical results.

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1 Introduction

In [3] we studied classes of multivariate Haar series f to the base $b \geq 2$, called ${}_b\tilde{E}_s^\alpha(C)$-classes, and their numerical integration using (t, m, s)-nets $\mathcal{P} = (\mathbf{x}_n)_{n=0}^{N-1}$ to the base b as node sets for the integration rule

$$\int_{[0,1[^s} f(\mathbf{x})\, d\mathbf{x} \approx \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n). \tag{1.1}$$

Sobol' [10] already studied special classes of Haar series to the base 2 with respect to numerical integration. Searching an optimal node set to integrate his classes, he defined the (t, m, s)-nets to base 2 (originally called P_τ-nets). Niederreiter generalized the (t, m, s)-nets to arbitrary bases and gave efficient construction methods and an exhaustive theory (e.g. see [8]).

The present paper completes the studies in [3] in the sense that we compare the different classes of multivariate Haar series that have been studied with respect to numerical integration. In Section 3 we will consider Sobol's S_p- and H_α-classes (the S_p-classes will be defined for arbitrary integer bases $b \geq 2$) and compare them to our ${}_b\tilde{E}_s^\alpha(C)$-classes. The S_p- and ${}_b\tilde{E}_s^\alpha(C)$-classes are defined using resolution dependence only. Hence the basic ideas in Section 3 probably may be extended to other wavelet systems on compact intervals. Section 4 contains a comparison of the integration error estimates for these classes and numerical results. Finally, in Section 5 we will give supplementing properties of ${}_b\tilde{E}_s^\alpha(C)$-classes.

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2 Generalized Haar function systems

In the following we fix an arbitrary integer base $b \geq 2$. For the notations and definitions of generalized Haar functions relative to base b, see [3, 4]. In this section we recall the basic notations:

For an integer $k \geq 0$ and an arbitrary number $x \in [0,1[$, let $k = \sum_{j=0}^{\infty} k_j b^j$ and $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, $k_j, x_j \in \{0,1,\ldots,b-1\}$, be the b-adic expansions of k and x in base b. For $g \in \mathbf{N}$ we define $k(g) := \sum_{j=0}^{g-1} k_j b^j$ and $x(g) := \sum_{j=0}^{g-1} x_j b^{-j-1}$. Further let $k(0) := 0$ and $x(0) := 0$. The support of a given Haar function h_k, $k \geq 0$, is equal to an elementary b-adic interval. We now define sets of integers k, for which such intervals have the same length (resolution).

Definition 2.1.
(1) Let g be a nonnegative integer. Then $\Delta(g) := \{k \in \mathbf{N} : b^g \leq k < b^{g+1}\}$. Further, let $\Delta(-1) := \{0\}$, and the sets $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, and $\mathbf{N}_1 = \mathbf{N}_0 \cup \{-1\}$.
(2) If $\mathbf{g} = (g_1, \ldots, g_s)$, $s \geq 2$ and $g_i \in \mathbf{N}_1$, then $\Delta(\mathbf{g}) := \prod_{i=1}^{s} \Delta(g_i)$.

Definition 2.2.
Let $e_b : \mathbf{Z}_b \to \mathbf{K}$, where $\mathbf{Z}_b = \{0, \ldots, b-1\}$ is the least residue system modulo b, and $\mathbf{K} := \{z \in \mathbf{C} : |z| = 1\}$, denote the function $e_b(a) := \exp(2\pi i \frac{a}{b})$, $(a \in \mathbf{Z}_b)$. The k-th Haar function h_k, $k \geq 0$, to the base b is defined as follows: If $k = 0$, then $h_0(x) := 1 \quad \forall x \in [0,1[$. If $k \in \Delta(g)$, $g \geq 0$, then

$$h_k(x) := b^{\frac{g}{2}} \cdot \sum_{a=0}^{b-1} e_b(a \cdot k_g) \cdot \mathbf{1}_{D_k(a)}(x),$$

with elementary b-adic intervals $D_k(a) := [(b\,k(g)+a)/b^{g+1}, (b\,k(g)+a+1)/b^{g+1}[$. The k-th normalized Haar function H_k on $[0,1[$ is defined as $H_0 := h_0$ and, if $k \in \Delta(g)$, $g \geq 0$, then $H_k := b^{-\frac{g}{2}} \cdot h_k$.

Hence, the support D_k of the k-th Haar function h_k is given as the following elementary b-adic interval : If $k = 0$, then $D_0 := [0,1[$. If $k \in \Delta(g)$, $g \geq 0$, then $D_k := \bigcup_{a=0}^{b-1} D_k(a) = [k(g)/b^g, (k(g)+1)/b^g[$.

Definition 2.3. *Let $\mathcal{H}_b := \{h_{\mathbf{k}} : \mathbf{k} := (k_1, \ldots, k_s) \in \mathbf{N}_0^s\}$ denote the Haar function system to the base b on the s-dimensional torus $[0,1[^s$, $s \geq 1$. The $\mathbf{k}$-th Haar function $h_{\mathbf{k}}$ is defined as $h_{\mathbf{k}}(\mathbf{x}) := \prod_{i=1}^{s} h_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1[^s$. The normalized version $H_{\mathbf{k}}$ is defined in the same way and the supports of $h_{\mathbf{k}}$ and $H_{\mathbf{k}}$ are defined as $D_{\mathbf{k}} := \prod_{i=1}^{s} D_{k_i}$.*

3 Classes of multivariate Haar series

Consider $f \in L^1([0,1[^s, \lambda_s)$, where λ_s denotes the Lebesgue measure on the s-dimensional unit cube $[0,1[^s$, $s \geq 2$. For $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbf{N}_0^s$ let r be the number of components k_i with $k_i \geq 1$ ($k_j = 0$ for $j \neq i$). Therefore a Haar series s_f of f has the form

$$(3.1) \quad s_f(\mathbf{x}) := \hat{f}(\mathbf{0}) + \sum_{r=1}^{s} \sum_{1 \leq i_1 < \ldots < i_r \leq s} \sum_{k_{i_j}=1}^{\infty} \hat{f}_r(\mathbf{k}) \cdot h_{k_{i_1}}(x_{i_1}) \cdots h_{k_{i_r}}(x_{i_r})$$

with Haar coefficients

$$\hat{f}_r(\mathbf{k}) := \int_{[0,1[^s} f(\mathbf{x}) \cdot \overline{h_{k_{i_1}}(x_{i_1})} \cdots \overline{h_{k_{i_r}}(x_{i_r})}\, d\mathbf{x} \tag{3.2}$$

Let $f \equiv s_f$ and s_f absolutely convergent. Using a point set $\mathcal{P} = (\mathbf{x}_n)_{n=0}^{N-1}$, $\mathbf{x}_n = (x_1^{(n)}, \ldots, x_s^{(n)}) \in [0,1[^s$, in rule (1.1) to approximate the integral we obtain the following form of the integration error

$$R_N(f,\mathcal{P}) = \left| \sum_{r=1}^{s} \sum_{1 \le i_1 < \ldots < i_r \le s} \sum_{k_{i_j}=1}^{\infty} \hat{f}_r(\mathbf{k}) \cdot \tfrac{1}{N} \sum_{n=0}^{N-1} h_{k_{i_1}}(x_{i_1}^{(n)}) \cdots h_{k_{i_r}}(x_{i_r}^{(n)}) \right|$$

$$\le \sum_{r=1}^{s} \sum_{1 \le i_1 < \ldots < i_r \le s} \sum_{g_{i_j}=0}^{\infty} b^{\frac{1}{2}(g_{i_1}+\ldots+g_{i_r})} \sum_{k_{i_j} \in \Delta(g_{i_j})} |\hat{f}_r(\mathbf{k})| \cdot |S_N^{(r)}(H_{\mathbf{k}},\mathcal{P})|, \tag{3.3}$$

with

$$S_N^{(r)}(H_{\mathbf{k}},\mathcal{P}) := \frac{1}{N} \sum_{n=0}^{N-1} H_{k_{i_1}}(x_{i_1}^{(n)}) \cdots H_{k_{i_r}}(x_{i_r}^{(n)}). \tag{3.4}$$

It follows that the point set $\mathcal{P}$ enters the integration error only with the order of magnitude of the "Weyl" sums $S_N^{(r)}(H_{\mathbf{k}},\mathcal{P})$, hence depends only on the Haar function system. The so called (t,m,s)-nets to the base b are point sets $\mathcal{P}$, consisting of $N = b^m$ points, which satisfy certain local conditions suitable to the local definition of the Haar functions. It follows that the Weyl sums vanish for a large set of resolutions $\mathbf{g}$ near the origin, more precisely

$$S_N(h_{\mathbf{k}},\mathcal{P}) = 0 \quad \text{for all} \quad \mathbf{k} \in \Delta(\mathbf{g}) \quad \text{and} \quad \mathbf{g} \quad \text{with} \quad \sum_{i=1}^{s}(g_i+1) \le m-t.$$

Further, for arbitrary (t,m,s)-nets to the base b for the remaining resolutions it follows that $|S_N(H_{\mathbf{k}},\mathcal{P})| \le b^{-(g_{i_1}+\ldots+g_{i_r})}$ for all $\mathbf{k} \in \Delta(\mathbf{g})$, and $\mathbf{g}$ with $m-t < \sum_{i=1}^{s}(g_i+1) < m-t+r$, and if $\sum_{i=1}^{s}(g_i+1) \ge m-t+r$, then there are at most $(b-1)^r b^m$ vectors $\mathbf{k} \in \Delta(\mathbf{g})$ with $|S_N(H_{\mathbf{k}},\mathcal{P})| \ne 0$. In the latter case we have $|S_N(H_{\mathbf{k}},\mathcal{P})| \le b^{t-m}$ (see [3, Lemma 4.2]). For special construction methods these estimates may be improved. Examples are given in [2]. For the general definition of (t,m,s)-nets and efficient construction methods see [1, 6, 8].

The main goal in numerical integration is to find large classes of functions which guarantee best possible integration errors. If we, for example, consider classes of multivariate Haar series with in a certain sense bounded Haar coefficients we are able to estimate the integration error (3.3). In [3] we observed Haar series where the Haar coefficients above are bounded in the following way

$$|\hat{f}_r(\mathbf{k})| \le C \cdot b^{-\alpha(g_{i_1}+\ldots+g_{i_r})} =: C \cdot \delta_b(\mathbf{k}). \tag{3.5}$$

Haar series with this property are called ${}_b\tilde{E}_s^{\alpha}(C)$-classes. For the exact definition and properties see [3, Sect. 2.2]. This approach is due to Korobov [5] who studied multivariate Fourier series. We changed Korobov's condition slightly in order to get a resolution dependency only.

Sobol' [10] studied more general classes called S_p, $p \geq 1$, classes. We will define these classes for arbitrary bases $b \geq 2$. Let $1 \leq p < \infty$ and q such that $1/p + 1/q = 1$. Applying Hölder's inequality in (3.3),

$$\sum_{k_{i_j} \in \Delta(g_{i_j})} |\hat{f}_r(\mathbf{k})| \cdot |S_N^{(r)}(H_{\mathbf{k}}, \mathcal{P})| \leq \left[\sum_{k_{i_j} \in \Delta(g_{i_j})} |\hat{f}_r(\mathbf{k})|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{k_{i_j} \in \Delta(g_{i_j})} |S_N^{(r)}(H_{\mathbf{k}}, \mathcal{P})|^q \right]^{\frac{1}{q}},$$

yields a more general estimate of $R_N(f, \mathcal{P})$. This estimate suggests the following definition, see also [10, p. 133].

DEFINITION 3.1. *For* $1 \leq p < \infty$ *let*

$$A_p^{(i_1,\ldots,i_r)}(s_f) := \sum_{g_{i_j}=0}^{\infty} b^{\frac{1}{2}(g_{i_1}+\ldots+g_{i_r})} \left[\sum_{k_{i_j} \in \Delta(g_{i_j})} |\hat{f}_r(\mathbf{k})|^p \right]^{\frac{1}{p}}. \tag{3.6}$$

A Haar series s_f *belongs to the class* $S_p^{(b)}(A_{i_1,\ldots,i_r})$ *if* $A_p^{(i_1,\ldots,i_r)}(s_f) \leq A_{i_1,\ldots,i_r}$, *for upper bounds* $A_{i_1,\ldots,i_r} > 0$ *(which only depend on the resolutions* $g_{i_1}, \ldots, g_{i_r}$*) and for all* $1 \leq i_1 < \ldots < i_r \leq s$, $1 \leq r \leq s$.

If $s_f \in S_p^{(b)}(A_{i_1,\ldots,i_r})$ then s_f converges absolutely and uniformly (see [10, p. 134]). Further, for $1 < p < p'$, we have $S_1^{(b)}(A_{i_1,\ldots,i_r}) \subset S_p^{(b)}(A_{i_1,\ldots,i_r}) \subset S_{p'}^{(b)}(A_{i_1,\ldots,i_r})$. The next proposition shows the relation between ${}_b\tilde{E}_s^{\alpha}(C)$- and $S_p^{(b)}$-classes.

PROPOSITION 3.1. *If* $\beta := \alpha - \frac{1}{2} - \frac{1}{p} > 0$, *then*

$${}_b\tilde{E}_s^{\alpha}(C) \subset S_p^{(b)}(A_{i_1,\ldots,i_r}) \quad \text{for} \quad A_{i_1,\ldots,i_r} = C \cdot (b-1)^{\frac{r}{p}} \cdot \left(\frac{b^{\beta}}{b^{\beta}-1} \right)^r$$

The proof is verified easily if we estimate $A_p^{(i_1,\ldots,i_r)}(s_f)$ using (3.5). The inclusion in Proposition 3.1 is strong. For example let $s = 1$ and $A > 0$. The Haar series s_f with

$$\hat{f}(k) := \begin{cases} \frac{A}{b^{\frac{g}{2}}} \cdot \frac{6}{\pi^2 g^2} & : \quad k = b^g, g \geq 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

belongs to $S_p^{(b)}(A)$ and not to any class ${}_b\tilde{E}_s^{\alpha}(C)$ for $\alpha > 1/2 + 1/p$.

Sobol' also studied classes $H_{\bar{\alpha}}(L_{i_1,\ldots,i_r})$. Let $s = 1$. A class $H_{\bar{\alpha}}(L)$, $0 < \bar{\alpha} \leq 1$, $L > 0$ consists of all functions $f : [0,1[\longrightarrow \mathbf{R}$, where $\forall\, x, y \in [0,1[$: $|f(x) - f(y)| \leq L|x-y|^{\bar{\alpha}}$. The multidimensional case $H_{\bar{\alpha}}(L_{i_1,\ldots,i_r})$, which differs from general Hölder - classes, is defined in [10, p. 136]. From [10, p. 141: (4.25)] we obtain

$$\begin{aligned} |\hat{f}_r(\mathbf{k})| &\leq L_{i_1,\ldots,i_r} \prod_{j=1}^{r} 2^{-(g_{i_j}+1)(\bar{\alpha}+\frac{1}{2})-\frac{1}{2}} \\ &\leq \frac{\max_{i_1,\ldots,i_r} L_{i_1,\ldots,i_r}}{2^{(\bar{\alpha}+1)}} \cdot \frac{1}{2^{(\bar{\alpha}+\frac{1}{2})(g_{i_1}+\ldots+g_{i_r})}}, \end{aligned}$$

and therefore the following proposition.

PROPOSITION 3.2. *Let $0<\bar{\alpha}\leq 1$ and $s\geq 2$, then*

$$H_{\bar{\alpha}}(L_{i_1,\ldots,i_r})\subset {}_2\tilde{E}_s^{\bar{\alpha}+\frac{1}{2}}(D)\quad with\quad D:=\frac{\max_{i_1,\ldots,i_r}L_{i_1,\ldots,i_r}}{2^{(\bar{\alpha}+1)}}.$$

The inclusion again is strong. An example of our ${}_2\tilde{E}_s^{\alpha}(C)$-classes which does not belong to $H_{\bar{\alpha}}(L_{i_1,\ldots,i_r})$ is the step function (1) in [3, Sect. 2.3].

4 Estimates of the integration error and numerical results

In this section we compare integration error estimates for the different classes of Sect. 3 if we use (t,m,s)-nets to calculate the integral approximation (1.1).

THEOREM 4.1. *Let $\mathcal{P}=\{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}$ be a (t,m,s)-net to the base $b\geq 2$.*

(a) *If $f\in {}_b\tilde{E}_s^{\alpha}(C)$ and $A:=\sum_{i=1}^{\infty}i^{s-1}b^{(\frac{1}{2}-\alpha)i}$, then*

$$R_N(f,\mathcal{P})\leq\frac{C\cdot A\cdot(1+2^{s-1})b^{(\alpha+\frac{1}{2})s}}{(\log b)^{s-1}}\cdot b^{(\alpha+\frac{1}{2})\cdot t}\cdot\frac{(\log N)^{s-1}}{N^{(\alpha-\frac{1}{2})}}.$$

(b) *If $f\in S_p^{(b)}(A_{i_1,\ldots,i_r})$ and q with $1/p+1/q=1$, then*

$$R_N(f,\mathcal{P})\leq(b-1)^{\frac{s}{q}}\cdot(b^t+b^{\frac{t+s}{p}})\cdot A'\cdot\frac{1}{N^{\frac{1}{p}}},\quad A':=\sum_{r=1}^{s}\sum_{1\leq i_1<\ldots<i_r\leq s}A_{i_1,\ldots,i_r}.$$

REMARK 4.1. Note that our estimates for $S_p^{(b)}(A_{i_1,\ldots,i_r})$-classes are valid for arbitrary (t,m,s)-nets in base b and they are more precise than those of Sobol' (see [10, p. 228,239]). Using P_τ-nets to integrate $H_{\bar{\alpha}}(L_{i_1,\ldots,i_r})$-classes, Sobol' [10, p. 239] obtained an estimate of the integration error in the same order of magnitude as in (a). The estimate in (a) is best possible for ${}_b\tilde{E}_s^{\alpha}(C)$-classes [2].

PROOF. The proof of Part (a) is given in [3, Part (c), Sect. 4.1]. To prove Part (b) we apply Hölder's inequality in (3.3). Then, from the estimates of the Weyl sums given above, we obtain $R_N(f,\mathcal{P})\leq$

$$\sum_{r=2}^{s}\sum_{1\leq i_1<\ldots<i_r\leq s}\sum_{\substack{g_{i_1},\ldots,g_{i_r}=0\\ m-t-r<g_{i_1}+\ldots+g_{i_r}<m-t}}^{m-t-1}(b-1)^{\frac{r}{q}}\cdot b^{(\frac{1}{2}-\frac{1}{p})(g_{i_1}+\ldots+g_{i_r})}\left[\sum_{k_{i_j}\in\Delta(g_{i_j})}|\hat{f}_r(\mathbf{k})|^p\right]^{\frac{1}{p}}$$

$$+\sum_{r=1}^{s}\sum_{1\leq i_1<\ldots<i_r\leq s}\sum_{\substack{g_{i_1},\ldots,g_{i_r}=0\\ g_{i_1}+\ldots+g_{i_r}\geq m-t}}^{\infty}(b-1)^{\frac{r}{q}}\cdot b^{(t-\frac{m}{p})}\cdot b^{\frac{1}{2}(g_{i_1}+\ldots+g_{i_r})}\left[\sum_{k_{i_j}\in\Delta(g_{i_j})}|\hat{f}_r(\mathbf{k})|^p\right]^{\frac{1}{p}}$$

For $f\in S_p^{(b)}(A_{i_1,\ldots,i_r})$ it follows that

$$\begin{aligned}R_N(f,\mathcal{P})\quad&\leq\quad b^{-m/p}\sum_{r=2}^{s}(b-1)^{r/q}\cdot b^{(t+r)/p}\sum_{1\leq i_1<\ldots<i_r\leq s}A_{i_1,\ldots,i_r}\\&+\quad b^{-m/p}\sum_{r=1}^{s}(b-1)^{r/q}\cdot b^{t}\sum_{1\leq i_1<\ldots<i_r\leq s}A_{i_1,\ldots,i_r},\end{aligned}$$

and this yields the result. □

As a numerical example we consider the function $F : [0,1[^s \longrightarrow \mathbf{R}$, with

$$F(x_1, \ldots, x_s) := (x_1 + \ldots + x_s)^n, \; n \in \mathbf{N}. \tag{4.1}$$

This function belongs to ${}_2\tilde{E}_s^{\frac{3}{2}}(C)$-classes. Using digital (t,m,s)-nets $\mathcal{P}$ from the Salzburg Tables [6] to calculate the approximation (1.1), we obtained the results in Figure 4.1 for the integration error $R_N(F,\mathcal{P})$, $N = 2^{23}$. The polynomials providing the nets and the corresponding quality parameters t are given in [6, Table 2]. The graphics demonstrates the behavior of the error (logarithmic scale in base 10) for increasing n and dimensions $3 \le s \le 15$.

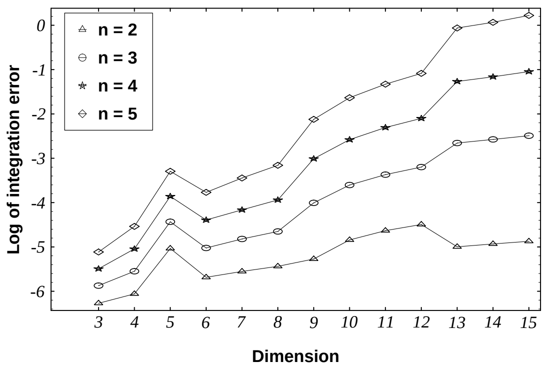

Figure 4.1: Integration errors with (t,m,s)-nets.

5 Further results for ${}_b\tilde{E}_s^{\alpha}(C)$-classes

In this section we present two results for ${}_b\tilde{E}_s^{\alpha}(C)$-classes which supplement the studies given in [3]. From a basic property of digital (t,m,s)-nets we immediately obtain the following result (compare also to [7, Thm. 2 (b)]).

Proposition 5.1. *For all $\alpha > 1/2$, $C > 0$ and all b and $s \ge 2$ we have: For all t and m for which there exists a digital (t,m,s)-net in base b, there is such a net and a function $f \in {}_b\tilde{E}_s^{\alpha}(C)$ with*

$$R_N(f,\mathcal{P}) = (b-1) \cdot \frac{b^{(\alpha-\frac{1}{2})}}{b^{(\alpha-\frac{1}{2})} - 1} \cdot b^{(\alpha-\frac{1}{2})t} \cdot \frac{1}{N^{(\alpha-\frac{1}{2})}}.$$

Proof. From the proof of [9, Lemma 3c] we get the following property of digital (t,m,s)-nets: For all t and m for which there exists a digital (t,m,s)-net in base b, there is such a net $\mathcal{P}_t$ where the first $m-t$ rows of $C^{(1)}$ equal the first $m-t$ rows of the identity matrix and the remaining rows are zero vectors. Therefore the first coordinates $x_1^{(n)}$, $0 \le n < b^m$, of $\mathcal{P}_t$ cover the set $\{0, 1/b^{m-t}, \ldots, (b^{m-t}-1)/b^{m-t}\}$.

Consider the Haar series $\sum_{\mathbf{k}\in\mathbf{N}_0^s} \hat{f}(\mathbf{k}) \cdot h_{\mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in [0,1[^s$ where

$$\hat{f}(\mathbf{k}) := \begin{cases} \frac{C}{b^{\alpha g}} & : \quad \mathbf{k} \in \Delta(\mathbf{g}),\ \mathbf{g} = (g,-1,\ldots,-1),\ g \geq 0 \\ 0 & : \quad \text{otherwise} \end{cases} .$$

For this function we get

$$R_N(f,\mathcal{P}_t) = \left| \sum_{g=0}^{\infty} \sum_{k\in\Delta(g)} \frac{C}{b^{(\alpha-\frac{1}{2})g}} \cdot \frac{1}{N} \sum_{n=0}^{N-1} H_k(x_1^{(n)}) \right| .$$

From [3, Lemma 4.1] we obtain the result. □

In the following we will observe [3, Thm. 3.2] in reverse direction. The result is an analogue to [7, Lemma 8] where generalized Walsh functions are used instead of Haar functions. Note that in our case there are no further limitations on the parameters $\alpha > 1/2$, due to the local structures of Haar functions.

THEOREM 5.2. *If* $f \in {}_B\tilde{E}_s^{\alpha}(C)$ *with* $B = b^L$, $L \geq 2$, *and* $\alpha > 1/2$, *then*

$$f \in {}_b\tilde{E}_s^{\alpha}(C \cdot C(\alpha,b)^s) \quad \textit{where} \quad C(\alpha,b) := \frac{B^{(\alpha+\frac{1}{2})}}{b^{(\alpha+\frac{1}{2})}} \cdot \frac{1}{B} \sum_{a=1}^{B-1} \frac{1}{\sin\ \pi\frac{a}{B}}.$$

PROOF. Let $B = b^L$, $L \geq 2$, for a given base $b \geq 2$. The result is proven by induction in a similar way as [3, Thm. 3.2]. To avoid too many indices, we shall denote the Haar coefficients of a given function f with respect to the system $\mathcal{H}_B$ by $\hat{f}(n)$, hence with argument n. Further we will use the argument k to signify that $\hat{f}(k)$ denotes the Haar coefficient of f with respect to the Haar system $\mathcal{H}_b$. We start with dimension $s = 1$.

For $B^{\bar{g}} \leq n < B^{\bar{g}+1}$, $\bar{g} \in \mathbf{N}_0$, the Haar function $h_n^{(B)} \in \mathcal{H}_B$ has a finite Haar series with respect to the system $\mathcal{H}_b$, more precisely

$$h_n^{(B)}(x) = \sum_{k=B^{\bar{g}}}^{B^{\bar{g}+1}-1} \widehat{h_n^{(B)}}(k) \cdot h_k(x). \tag{5.1}$$

Therefore, if $f \in {}_B\tilde{E}_1^{\alpha}(C)$, $\alpha > 1/2$, we obtain the following form of $\hat{f}(k)$,

$$\hat{f}(k) = \sum_{n=B^{\bar{g}}}^{B^{\bar{g}+1}-1} \hat{f}(n) \cdot \widehat{h_n^{(B)}}(k) \qquad \text{for} \qquad B^{\bar{g}} \leq k < B^{\bar{g}+1}. \tag{5.2}$$

Consider integers n,k with $B^{\bar{g}} \leq n < B^{\bar{g}+1}$, $\bar{g} \geq 0$ and $k \in \Delta(g)$ with $g = L\bar{g} + j, \quad 0 \leq j < L$. Further let us define

$$E_a := D_k(a) = [\beta^{(a)}, \beta^{(a+1)}[\quad \text{with} \quad \beta^{(a)} := \frac{k(g)}{b^g} + \frac{a}{b^{g+1}}, \quad 0 \leq a < b.$$

Therefore we have

$$\overline{\widehat{h_n^{(B)}}(k)} = b^{\frac{g}{2}} \sum_{a=0}^{b-1} e_b(a \cdot k_g) \cdot \widehat{\mathbf{1}_{E_a}}(n). \tag{5.3}$$

The latter result and (5.2) imply

$$|\hat{f}(k)| \leq b^{\frac{g}{2}} \cdot \frac{C}{B^{\bar{g}\alpha}} \cdot \sum_{n=B^{\bar{g}}}^{B^{\bar{g}+1}-1} \sum_{a=0}^{b-1} |\widehat{\mathbf{1}_{E_a}}(n)|. \tag{5.4}$$

From Hellekalek [4, Lemma 3.2], we observe that we may have $\widehat{\mathbf{1}_{E_a}}(n) \neq 0$ only if $n(\bar{g}) \in \{B^{\bar{g}} \cdot \beta^{(a)}(\bar{g}), B^{\bar{g}} \cdot \beta^{(a+1)}(\bar{g})\}$, and in this case it follows that $|\widehat{\mathbf{1}_{E_a}}(n)| \leq B^{(-\frac{\bar{g}}{2}-1)}/\sin \pi \frac{n_{\bar{g}}}{B}$. Hence we have to analyze for which $B^{\bar{g}} \leq n < B^{\bar{g}+1}$ and $0 \leq a \leq b$ the equation $n(\bar{g}) = B^{\bar{g}} \cdot \beta^{(a)}(\bar{g})$ holds. Comparing the B-adic expansions of $n(\bar{g})$ and $B^{\bar{g}} \cdot \beta^{(a)}(\bar{g})$ implies that for arbitrary $a \in \{0, \ldots, b\}$ there are $B-1$ such numbers n ($n(\bar{g})$ is fixed and $n_{\bar{g}}$ varies in $\{1, \ldots, B-1\}$). Using (5.4) we observe that

$$|\hat{f}(k)| \leq C \cdot B^{-(\alpha-\frac{1}{2})\bar{g}} \cdot b^{(\frac{g}{2}+1)} \cdot \frac{1}{B} \sum_{j=1}^{B-1} \frac{1}{\sin \pi \frac{j}{B}} \tag{5.5}$$

$$\leq C \cdot C(\alpha, b) \cdot \frac{1}{b^{\alpha g}}, \tag{5.6}$$

and this yields the result for dimension $s = 1$. As an induction hypothesis, we assume that $f \in {}_B\tilde{E}^{\alpha}_{s-1}(C)$ implies $f \in {}_b\tilde{E}^{\alpha}_{s-1}(C \cdot C(\alpha, b)^{s-1})$. Therefore

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{N}_0^{s-1}} \hat{f}_b(\mathbf{k}) \cdot h_{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbf{N}_0^{s-1}} \hat{f}_B(\mathbf{n}) \cdot h_{\mathbf{n}}^{(B)}(\mathbf{x}), \quad \mathbf{x} \in [0,1[^{s-1},$$

with $|\hat{f}_b(\mathbf{k})| \leq C \cdot C(\alpha, b)^{s-1} \cdot \delta_b(\mathbf{k})$ and $|\hat{f}_B(\mathbf{n})| \leq C \cdot \delta_B(\mathbf{n})$. Let $f \in {}_B\tilde{E}^{\alpha}_s(C)$. This yields $|\hat{f}_B(n_1, \ldots, n_s)| \leq C \cdot \delta_B(n_1) \cdot \delta_B(\mathbf{n})$, $\mathbf{n} = (n_2, \ldots, n_s)$, and therefore

$$f(x_1, \ldots, x_s) = \sum_{n_1=0}^{\infty} \underbrace{\left(\sum_{\mathbf{n} \in \mathbf{N}_0^{s-1}} \hat{f}_B(\mathbf{n}) \cdot h_{\mathbf{n}}^{(B)}(\mathbf{x}) \right)}_{\in {}_B\tilde{E}^{\alpha}_{s-1}(C \cdot \delta_B(n_1))} \cdot h_{n_1}^{(B)}(x_1), \;\; \mathbf{x} = (x_2, \ldots, x_s).$$

By induction, we get

$$f(x_1, \ldots, x_s) = \sum_{n_1=0}^{\infty} \left(\sum_{\mathbf{k} \in \mathbf{N}_0^{s-1}} \hat{f}_b^{(n_1)}(\mathbf{k}) \cdot h_{\mathbf{k}}(\mathbf{x}) \right) \cdot h_{n_1}^{(B)}(x_1),$$

with $|\hat{f}_b^{(n_1)}(\mathbf{k})| \leq C \cdot \delta_B(n_1) \cdot C(\alpha, b)^{s-1} \cdot \delta_b(\mathbf{k})$. On the other hand

$$f(x_1, \ldots, x_s) = \sum_{\mathbf{k} \in \mathbf{N}_0^{s-1}} h_{\mathbf{k}}(\mathbf{x}) \underbrace{\sum_{n_1=0}^{\infty} \hat{f}_b^{(n_1)}(\mathbf{k}) \cdot h_{n_1}^{(B)}(x_1)}_{\in_B \tilde{E}_1^{\alpha}(C \cdot \delta_b(\mathbf{k}) \cdot C(\alpha, b)^{s-1})} .$$

Applying the result for dimension $s = 1$ finishes the proof. □

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