BIT 37:(4) (1997), 845–860.

QUASI-MONTE CARLO METHODS FOR NUMERICAL INTEGRATION OF MULTIVARIATE HAAR SERIES

KARL ENTACHER *

Department of Mathematics, University of Salzburg, Hellbrunnerstr. 34
A-5020 Salzburg, Austria. email: Karl.Entacher@sbg.ac.at

Abstract.

In the present paper we study quasi-Monte Carlo methods to integrate functions representable by generalized Haar series in high dimensions. Using (t, m, s)-nets to calculate the quasi-Monte Carlo approximation, we get best possible estimates of the integration error for practically relevant classes of functions. The local structure of the Haar functions yields interesting new aspects in proofs and results. The results are supplemented by concrete computer calculations.

AMS subject classification: Primary 65D30, 42C10.

Key words: Numerical integration, generalized Haar functions, low-discrepancy point sets, quasi-Monte Carlo methods.

1 Introduction

Quasi-Monte Carlo methods are the most effective approach for the approximate calculation of high dimensional integrals. We refer the reader to the comprehensive monograph of Niederreiter [12].

The basic tools for these methods are special low-discrepancy point sets $\mathcal{P} = (\mathbf{x}_n)_{n=0}^{N-1}$, $N \in \mathbf{N}$, in the s-dimensional unit cube $[0,1[^s$, $s \geq 2$. The quasi-Monte Carlo approximation of an integral of a function $f : [0,1[^s \longrightarrow \mathbf{R}$ is given by

$$\int_{[0,1[^s} f(\mathbf{x})\, d\mathbf{x} \approx \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n).$$

Optimal estimates of the integration error

$$R_N(f, \mathcal{P}) = \left| \int_{[0,1[^s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \tag{1.1}$$

are obtained for classes of functions representable by special orthogonal series and for suitable point sets. Using special classes of rapidly converging Fourier

*Research supported by the Austrian Science Foundation (FWF), project no. P9285, P11143-MAT, and by CEI-PACT, WP 5.1.2.1.2.

series called E_s^α classes, Korobov[4] developed the theory of good lattice points. Recently, in a series of papers, Larcher et al. [6, 7, 8, 9] studied Korobov's approach using generalized Walsh series. The appropriate point sets in this study are Niederreiter's (t, m, s)-nets. The latter author has given a detailed theory on these low-discrepancy point sets and efficient construction methods (see [12]).

Historically, the definition of (t, m, s)-nets in base 2 is due to Sobol'[13, 14], who called them P_τ-nets. It was Sobol's goal to study quasi-Monte Carlo integration in terms of the classical Haar system. He mainly considered classes of functions which satisfy the Hölder condition, called H_α classes and special Haar series with in a certain sense bounded sum of the Haar coefficients (S_p classes).

Considering this background, the following questions naturally arise: If we study Korobov's approach using (t, m, s)-nets in terms of *generalized* Haar function systems, how efficient is this approach in comparison to Sobol's estimates and the estimates in the Walsh case, and how does the *local nature* of the Haar functions influence proofs and results?

This paper is devoted to an elaborate study of these two questions. Our approach yields best possible integration error estimates for (t, m, s)-nets as in the Walsh case. The integration errors for our $\tilde{E}^\alpha$ classes show the same order of magnitude as Sobol' derived for his H_α classes, which are subclasses of our classes. Analogous results to those of Larcher [7, 9] can be proven in an entirely different and sometimes easier way (compare our Theorem 3.2 and Theorem 3 in [7]) due to the local definition of the Haar functions. Contrary to the Walsh case it can be shown that practically relevant classes of functions satisfy the conditions required for the error estimates.

The results of this paper, which are part of the author's PhD thesis[1] are obtained by *multiresolution properties* of the Haar functions only. We conjecture that our concept may be extended to other orthogonal *wavelet* systems on compact intervals.

2 Definitions

2.1 The generalized Haar function system

In this section we present the definition of the Haar function system relative to an arbitrary integer base $b \geq 2$. The notations and the definitions are taken from Hellekalek [3].

In the following, we shall identify the s-dimensional torus $(\mathbf{R}/\mathbf{Z})^s$, $s \geq 1$, with the half-open unitcube $[0,1[^s$. The normalized Haar measure on $(\mathbf{R}/\mathbf{Z})^s$, respectively the Lebesgue measure on $[0,1[^s$, will be denoted by λ_s.

Let $b \geq 2$ be a fixed integer. For a nonnegative integer k, let $k = \sum_{j=0}^{\infty} k_j b^j$, $k_j \in \{0, 1, \ldots, b-1\}$, be the unique b-adic expansion of k in base b. Every number $x \in [0,1[$ has a unique b-adic expansion $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, $x_j \in \{0, 1, \ldots, b-1\}$, under the condition that $x_j \neq b-1$ for infinitely many j. In the following, this uniqueness condition is assumed without further notice.

For $g \in \mathbf{N}$, we define $k(g) := \sum_{j=0}^{g-1} k_j b^j$ and $x(g) := \sum_{j=0}^{g-1} x_j b^{-j-1}$. Further let $k(0) := 0$ and $x(0) := 0$. Note that $k(g) \in \{0, 1, \ldots, b^g - 1\}$ and $x(g) \in \{0, 1/b^g, \ldots, (b^g - 1)/b^g\}$.

DEFINITION 2.1. *(1) Let g be a nonnegative integer. We define $\Delta_b(g) := \{k \in \mathbf{N} : b^g \leq k < b^{g+1}\}$. Further, let $\Delta_b(-1) := \{0\}$ and the sets $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, and $\mathbf{N}_1 = \mathbf{N}_0 \cup \{-1\}$.*
(2) If $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbf{N}_1^s$, for an integer $s \geq 2$, then $\Delta_b(\mathbf{g}) := \prod_{i=1}^s \Delta_b(g_i)$.

REMARK 2.1. Throughout this paper, we will use the sets $\Delta_b(g)$. The reader should note that, if $k \in \Delta_b(g),\ g \geq 0$, then $k = k_0 + k_1 b + \ldots + k_g b^g = k(g) + k_g b^g$, with $k_g \in \{1, \ldots, b-1\}$. If $g \geq 0$, then $\sharp\Delta_b(g) = (b-1) \cdot b^g$, where $\sharp M$ denotes the number of elements of a given set M.

DEFINITION 2.2. *Let $e_b : \mathbf{Z}_b \to \mathbf{K}$, where $\mathbf{Z}_b = \{0, 1, \ldots, b-1\}$ is the least residue system modulo b and $\mathbf{K} := \{z \in \mathbf{C} : |z| = 1\}$, denote the function*

$$e_b(a) := e^{2\pi i \frac{a}{b}}, \quad a \in \mathbf{Z}_b.$$

The k-th Haar function $h_k^{(b)}$, $k \geq 0$, to the base b is defined as follows:
If $k = 0$, then $h_0^{(b)}(x) := 1 \quad \forall x \in [0, 1[$. If $k \in \Delta_b(g),\ g \geq 0$, then

$$h_k^{(b)}(x) := b^{\frac{g}{2}} \cdot \sum_{a=0}^{b-1} e_b(a \cdot k_g) \cdot \mathbf{1}_{D_k^{(b)}(a)}(x),$$

with the elementary b-adic intervals

$$D_k^{(b)}(a) := \left[\frac{k(g)}{b^g} + \frac{a}{b^{g+1}}, \frac{k(g)}{b^g} + \frac{a+1}{b^{g+1}} \right[.$$

The k-th normalized Haar function $H_k^{(b)}$ on $[0, 1[$ is defined as $H_0^{(b)} := h_0^{(b)}$, and, if $k \in \Delta_b(g),\ g \geq 0$, then $H_k^{(b)} := b^{-\frac{g}{2}} \cdot h_k^{(b)}$.

DEFINITION 2.3. *The fundamental domain $D_k^{(b)}$ of the k-th Haar function $h_k^{(b)}$ is defined as the following elementary b-adic interval: If $k = 0$, then $D_0^{(b)} := [0, 1[$; if $k \in \Delta_b(g),\ g \geq 0$, then*

$$D_k^{(b)} := \bigcup_{a=0}^{b-1} D_k^{(b)}(a) = \left[\frac{k(g)}{b^g}, \frac{k(g)+1}{b^g} \right[.$$

REMARK 2.2. The notation "fundamental domain" is not common usage. We write h_k, H_k, $\Delta(\mathbf{g})$ and D_k if it is clear from the context which base b is meant or if we present properties of $h_k^{(b)}$ and $H_k^{(b)}$ which are valid for all bases $b \geq 2$. This notation will be used in the multidimensional case defined below.

DEFINITION 2.4. *Let $\mathcal{H}_b := \{h_{\mathbf{k}}^{(b)} : \mathbf{k} := (k_1, k_2, \ldots, k_s) \in \mathbf{N}_0^s\}$ denote the Haar function system to the base b on the s-dimensional torus $[0, 1[^s,\ s \geq 1$. The $\mathbf{k}$-th Haar function $h_{\mathbf{k}}^{(b)}$ is defined as $h_{\mathbf{k}}^{(b)}(\mathbf{x}) := \prod_{i=1}^s h_{k_i}^{(b)}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1[^s$. The normalized version $H_{\mathbf{k}}^{(b)}$ is defined in the same way and the $\mathbf{k}$-th s-dimensional fundamental domain denotes $D_{\mathbf{k}}^{(b)} := \prod_{i=1}^s D_{k_i}^{(b)}$.*

REMARK 2.3. The main properties of the Haar functions are given in [3, Remark 2.1, 2.2]. The generalization of property (4) in Remark 2.1 of the latter paper is the following: there are exactly $(b-1)^r$ Haar functions $h_{\mathbf{k}}^{(b)}$, $\mathbf{k} \in \Delta_b(\mathbf{g})$, $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbf{N}_1^s$, where $s - r$ is the number of indices i such that $g_i = -1$, that have the same fundamental domain. In this case we get

$$H_{\mathbf{k}}^{(b)} = b^{-\delta} \cdot h_{\mathbf{k}}^{(b)}, \quad \text{with} \quad \delta = \tfrac{1}{2} \sum_{\substack{i=1 \\ g_i \neq -1}}^{s} g_i.$$

2.2 *The function class ${}_b\tilde{E}_s^{\alpha}(C)$*

We define our function classes ${}_b\tilde{E}_s^{\alpha}(C)$ slightly different from Larcher. Since the supports of the Haar functions h_k, $k \in \Delta(g)$, $g \geq 0$, are elementary b-adic intervals of length b^{-g}, we define the classes with respect to the resolution b^{-g}.

For $f \in L^1([0,1[^s, \lambda_s)$, let S_f denote the Haar series of f,

$$S_f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbf{N}_0^s} \hat{f}(\mathbf{k}) \cdot h_{\mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in [0,1[^s, \quad \text{with} \quad \hat{f}(\mathbf{k}) := \int_{[0,1[^s} f \cdot \overline{h_{\mathbf{k}}} \, d\lambda_s.$$

DEFINITION 2.5. *For a given integer base $b \geq 2$ and for $\alpha > 0$ and $C > 0$, let ${}_b\tilde{E}_s^{\alpha}(C)$ be the class of all functions $f \in L^1([0,1[^s, \lambda_s)$ with $f \equiv S_f$ on $[0,1[^s$ where the Haar coefficients $\hat{f}$ have the following property*

$$|\hat{f}(\mathbf{k})| \leq \frac{C}{r_b(\mathbf{k})^{\alpha}}, \quad \forall\, \mathbf{k} \in \Delta(\mathbf{g}), \quad \textit{and} \quad \mathbf{g} = (g_1, \ldots, g_s) \in \mathbf{N}_1^s, \tag{2.1}$$

with

$$r_b(\mathbf{k}) := \prod_{i=1}^{s} r_b(k_i) \quad \textit{and} \quad r_b(k_i) := \begin{cases} 1 & : \quad k_i = 0 \\ b^{g_i} & : \quad k_i \in \Delta(g_i), \ g_i \geq 0. \end{cases}$$

REMARK 2.4. If $f \in {}_b\tilde{E}_s^{\alpha}(C)$ and $\alpha > 1/2$, then the Haar series S_f is absolutely convergent (see [1]). Example 1 below indicates a Haar series which is divergent for all $x \in [0,1[^s$ if $\alpha = 1/2$.

2.3 *Examples of functions and their classes*

In this section, we present some examples of functions for different classes ${}_2\tilde{E}_s^{\alpha}(C)$. The proofs that the functions belong to these classes are given in Entacher[1][1].

[1]This paper is also available in the internet on the World-Wide-Web server: `http://random.mat.sbg.ac.at` (also accessible via ftp).

(1) Our test function to calculate the integration error in high dimensions is the function $F : [0,1[^s \longrightarrow \mathbf{R}$ with

$$F(\mathbf{x}) := \prod_{i=1}^{s} f(x_i), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1[^s, \quad \text{and} \tag{2.2}$$

$$f(x) := \frac{2^{(\alpha+\frac{1}{2})} - 1}{2^{\alpha}(2^{(\alpha-\frac{1}{2})} - 1)} \cdot \frac{1}{2^{g(\alpha-\frac{1}{2})}} \quad \text{for} \quad x \in \left[1 - \frac{1}{2^g}, 1 - \frac{1}{2^{g+1}}\right[, \quad g \geq 0.$$

F belongs to the class ${}_2\tilde{E}_s^{\alpha}(C)$ with $\alpha > 1/2$ and $C := \max\{1, (2^{\alpha} - \sqrt{2})^{-s}\}$.

(2) Consider the class $H_{\alpha}(L)$, $0 < \alpha \leq 1$ of functions $f : [0,1[\longrightarrow \mathbf{R}$, where $\forall\, x, y \in [0,1[\ : \ |f(x) - f(y)| \leq L|x-y|^{\alpha}$. These function classes and their generalization to dimension $s \geq 2$ have been studied by Sobol' [14]. It is easy to show that $H_{\alpha}(L) \subset {}_2\tilde{E}_1^{\alpha+\frac{1}{2}}(L/2)$.

(3) Examples of the classes ${}_2\tilde{E}_s^{\frac{3}{2}}(C)$.

From **(2)** we obtain that the function $f : [0,1[\longrightarrow \mathbf{R}$, $f(x) := x^n$, $n \in \mathbf{N}$, belongs to the class ${}_2\tilde{E}_1^{\frac{3}{2}}(\frac{n}{2})$. Therefore ${}_2\tilde{E}_1^{\frac{3}{2}}(|a_0| + \frac{1}{2}\sum_{i=1}^{n} i \cdot |a_i|)$ is the class of the polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad n \in \mathbf{N}_0, \quad a_i \in \mathbf{R},$$

and ${}_2\tilde{E}_s^{\frac{3}{2}}(\frac{1}{2}\sum_{i=1}^{s} i \cdot |a_i|)$ is the class of the function

$$F(x_1, \ldots, x_s) := a_1 x_1 + a_2 x_2^2 + \ldots + a_s x_s^s, \quad (x_1, \ldots, x_s) \in [0,1[^s.$$

Consider $F : [0,1[^s \longrightarrow \mathbf{R}$ with

$$F(x_1, \ldots, x_s) := (x_1 + \ldots + x_s)^n, \quad n \in \mathbf{N}. \tag{2.3}$$

This function belongs to the class ${}_2\tilde{E}_s^{\frac{3}{2}}(C)$ with

$$C = \sum_{\substack{n_1, \ldots, n_s = 0 \\ n_1 + \ldots + n_s = n}}^{n} \frac{n!}{2^{(s - \sharp\{n_i = 0\})} \cdot \prod_{i=1}^{s}(n_i - 1)!} \quad \text{where} \quad (-1)! := 1.$$

2.4 (t,m,s)-nets

In analogy to Larcher's method of numerical integration of Walsh series, we get optimal integration errors by using (t,m,s)-nets. For efficient construction methods of (t,m,s)-nets we refer to [5, 10, 11, 12]. The results of the integration error will be compared to the estimates derived by using the uniform lattice.

Special construction methods of (t,m,s)-nets, the so called digital nets, play an outstanding role in the Walsh case. First results concerning the use of digital nets in our framework are given in [1, 2].

The definition of (t,m,s)-nets and the basic properties are given in [12, p. 48].

DEFINITION 2.6.

Let $n \in \mathbf{N}$. The uniform lattice is defined as the following point set $\mathcal{P}$ consisting of $N = n^s$ points $\mathcal{P} := \{(\frac{k_1}{n}, \ldots, \frac{k_s}{n})\ :\ 0 \leq k_i \leq n-1,\ 1 \leq i \leq s\}$.

3 The results

3.1 Estimates of the integration error

THEOREM 3.1. *For $b \geq 2$, let $f \in {}_b\tilde{E}_s^\alpha(C)$ with $\alpha > \frac{1}{2}$. Further let $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be a point set in $[0,1[^s$ and $R_N(f,\mathcal{P})$ the integration error (1.1).*
(a) *If $\mathcal{P}$ is the uniform lattice with $N = b^{ns}$, $n \geq 1$, then*

$$R_N(f,\mathcal{P}) \leq C \cdot A \cdot b^{(\alpha-\frac{1}{2})}b^s \cdot \frac{1}{N^\gamma}, \qquad \gamma = \tfrac{\alpha-\frac{1}{2}}{s}, \qquad A := \sum_{i=1}^{\infty} i^{s-1} b^{(\frac{1}{2}-\alpha)i}.$$

(b) *The result in* (a) *is, apart of the constant C, best possible since there exists a function $f \in {}_b\tilde{E}_s^\alpha(1)$, with*

$$R_N(f,\mathcal{P}) = \frac{(b-1)b^{(\alpha-\frac{1}{2})}}{b^{(\alpha-\frac{1}{2})}-1} \cdot \frac{1}{N^\gamma}, \qquad \gamma = \tfrac{\alpha-1/2}{s}.$$

(c) *If $\mathcal{P}$ is a (t,m,s)-net to the base b, then*

$$R_N(f,\mathcal{P}) \leq \frac{C \cdot A \cdot (1+2^{s-1})b^{(\alpha+\frac{1}{2})s}}{(\log b)^{s-1}} \cdot b^{(\alpha+\frac{1}{2})\cdot t} \cdot \frac{(log\, N)^{s-1}}{N^{(\alpha-\frac{1}{2})}}.$$

(d) *The result in* (c) *is, apart of the constant, best possible, since for every $\alpha > \frac{1}{2}$ and $C > 0$ there exists a $(0,m,2)$-net $\mathcal{P}$ to the base 2 and a function $f \in {}_2\tilde{E}_2^\alpha(C)$ with*

$$R_N(f,\mathcal{P}) \geq C \cdot 2^{(\alpha-\frac{1}{2})} \cdot \frac{log\, N}{N^{(\alpha-\frac{1}{2})}}.$$

REMARK 3.1. An analogue to Theorem 3.1 for the Walsh case is given in Larcher and Traunfellner [9, Theorem 1 and Theorem 2]. Recent improvements of these results can be found in [8, Theorem 1 and Theorem 2]. The $(0,m,2)$-net in part (d) is an example of a digital $(t,m,2)$-net. Despite the fact that the error estimates above suggest to use $(0,m,s)$-nets to calculate the quasi-Monte Carlo approximation of a function, we can give examples of 2-dimensional functions belonging to our classes where, for example, the integration error using a $(1,m,2)$-net to calculate the quasi-Monte Carlo approximation is smaller than the error obtained by a $(0,m,2)$-net (see [1, 2]).

Using (t,m,s)-nets to base 2 (originally called P_τ-nets), Sobol' derived an integration error estimate for his H_α classes (compare Sect. 2.3 (2)) of the same order of magnitude as we obtained for our classes, see [14, p. 239]. Sobol's more general S_p classes exhibit higher error bounds.

Theorem 3.1 yields the best integration error estimate by using $(0,m,s)$-nets. But for $m \geq 2$, a $(0,m,s)$-net in base b can only exist for dimensions $s \leq b+1$ (see [12, Corollary 4. 21]). For applications on binary computers, efficient calculations are done with base 2, and for this reason, $(0,m,s)$-nets can only exist up to dimension $s = 3$.

We bridge this gap by the following theorem which shows that a function $f \in {}_b\tilde{E}_s^\alpha(C)$, $\alpha > 1/2$, belongs to a class ${}_{b^L}\tilde{E}_s^\alpha(D)$, $L \geq 2$. This guarantees a calculation of the quasi-Monte Carlo approximation with $(0, m, s)$-nets to the base b^L in higher dimensions.

THEOREM 3.2. *Let $f \in {}_b\tilde{E}_s^\alpha(C)$ with $\alpha > \frac{1}{2}$. Then we have*

$$f \in {}_B\tilde{E}_s^\alpha(C \cdot \bar{C}(\alpha, b)^s) \quad \textit{where} \quad \bar{C}(\alpha, b) = C(b) \cdot B \cdot \frac{b^{(\alpha+\frac{1}{2})}}{b^{(\alpha+\frac{1}{2})} - 1}$$

with

$$B = b^L,\ L \geq 2, \quad \textit{and} \quad C(b) = \frac{1}{b} \sum_{a=1}^{b-1} \frac{1}{\sin \pi \frac{a}{b}}.$$

REMARK 3.2. The analogue to Theorem 3.2 for the Walsh case (see Larcher et al. [7, Theorem 3]) is proven for the case $b = 2$. In comparison to our result, Larcher's theorem is only valid for $\alpha > 1 + \beta_L$, $0.25 \leq \beta_L < 0.5$, and the appropriate constant D contains B^α, whereas our constant D contains only B.

3.2 Numerical results

Here, we present some numerical results of the integration error (1.1) for the test function 2.2 described in Section 2.3 . Numerical results for function (2.3) are given in [2]. Further numerical comparisons for special Walsh series and the function $f(x_1, \ldots, x_s) := (x_1 + \ldots + x_s)^{1/2}$, using (t, m, s)-nets, good lattice points and Halton sequences are given in [6].

We normalized f, hence we get $\int_{[0,1[^s} F(\mathbf{x})\, d\mathbf{x} = 1$ in all dimensions. Using $\alpha = k + \frac{1}{2}$, $k \in \mathbf{N}$, we can calculate the quasi-Monte Carlo approximation for this function in integer arithmetic except of a final division. Thus, round off errors are avoided. The $(0, m, s)$-nets, $2 \leq s \leq 9$, are generated by the construction method published in [10].

	$\alpha = 3/2 \quad b = 4$				$\alpha = 7/2 \quad b = 4$			
$N \setminus s$	2	3	4	5	2	3	4	5
4^5	2.4e-05	3.6e-05	2.1e-04	1.2e-03	9.0e-10	7.9e-08	8.9e-06	1.9e-03
4^6	1.8e-06	6.1e-06	2.8e-05	2.9e-04	4.2e-12	4.7e-10	3.6e-08	2.3e-05
4^7	1.3e-07	2.3e-06	2.5e-06	5.1e-05	1.9e-14	1.3e-11	2.9e-11	2.1e-07
4^8	9.5e-09	8.4e-08	9.3e-08	1.1e-06	0.0	2.4e-14	6.0e-13	2.4e-10
4^9	6.7e-10	6.0e-10	1.1e-08	4.1e-07	0.0	0.0	4.2e-15	4.7e-12
	$\alpha = 3/2 \quad b = 8$				$\alpha = 7/2 \quad b = 8$			
$N \setminus s$	6	7	8	9	6	7	8	9
8^5	2.3e-05	2.0e-04	1.1e-03	6.8e-03	4.0e-08	1.5e-05	1.7e-04	3.4e-01
8^6	6.1e-07	5.9e-06	1.1e-05	3.9e-03	2.1e-10	1.4e-08	5.9e-06	3.4e-01
8^7	6.1e-08	4.1e-07	1.1e-05	3.9e-03	5.6e-13	3.3e-12	7.8e-08	3.4e-01
8^8	1.4e-09	1.7e-08	7.0e-08	1.9e-03	2.2e-16	7.2e-13	7.2e-12	3.0e-01
8^9	3.6e-12	1.3e-10	1.2e-09	3.1e-05	0.0	0.0	3.3e-14	7.3e-05

The following table presents some results using the uniform lattice in dimensions $3 \leq s \leq 8$.

$\alpha = 3/2 \quad b = 4$						$\alpha = 7/2 \quad b = 4$					
3		4		5		3		4		5	
N	R_N	N	R_N	N	R_N	N	R_N	N	R_N	N	R_N
4^9	3.7e-04	4^8	7.8e-03	4^{10}	9.8e-03	4^9	1.6e-07	4^8	5.6e-05	4^{10}	7.0e-05

$\alpha = 3/2 \quad b = 8$						$\alpha = 7/2 \quad b = 8$					
6		7		8		6		7		8	
8^6	4.8e-02	8^7	5.6e-02	8^8	6.4e-02	8^6	1.4e-03	8^7	1.6e-03	8^8	1.8e-03

4 The proofs

Let $f \in {}_b\tilde{E}_s^\alpha(C)$, $\alpha > 1/2$, and $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be a point set in $[0,1[^s$. Since $f = S_f$ on $[0,1[^s$, we easily get

$$R_N(f, \mathcal{P}) = \left| \sum_{\mathbf{k} \neq \mathbf{0}} \hat{f}(\mathbf{k}) \cdot S_N(h_{\mathbf{k}}, \mathcal{P}) \right|$$

with

$$S_N(h_{\mathbf{k}}, \mathcal{P}) := \frac{1}{N} \sum_{n=0}^{N-1} h_{\mathbf{k}}(\mathbf{x}_n), \quad \mathbf{k} \in \mathbf{N}_0^s.$$

From Definition 2.5, it follows

$$R_N(f, \mathcal{P}) \leq C \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{r_b(\mathbf{k})^\alpha} |S_N(h_{\mathbf{k}}, \mathcal{P})|. \tag{4.1}$$

In the following, we estimate the "Weyl sums" $S_N(h_{\mathbf{k}}, \mathcal{P})$ for the different point sets.

4.1 Proof of Theorem 3.1

For a given $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbf{N}_1^s$, let r, $1 \leq r \leq s$ denote the number of i with $g_i \geq 0$, and let the remaining $g_i = -1$ (i.e. $k_i = 0$). For the calculations below, the order in $(g_1, \ldots, g_s)$ does not matter. We only have to note that there are $\binom{s}{r}$ possibilities to arrange the $s - r$ numbers -1 in $\mathbf{g} = (g_1, \ldots, g_s)$. Therefore let, w. l. o. g., $\mathbf{g} \in \mathbf{N}_1^s$ with

$$g_i \geq 0 \ \text{ for } \ 1 \leq i \leq r \quad \text{and} \quad g_i = -1 \ \text{ for } \ r+1 \leq i \leq s. \tag{4.2}$$

Part (a) Let $\mathcal{P}$ be the uniform lattice with $N = b^{ns}$ points.

LEMMA 4.1.
(1) *$S_N(h_{\mathbf{k}}, \mathcal{P}) = 0$ for all $\mathbf{k} \in \Delta(\mathbf{g})$ and $\mathbf{g}$ with at least one $g_j < n$ for $1 \leq j \leq r$.*
(2) *For all $\mathbf{g}$ with $g_j \geq n$, $1 \leq j \leq r$, we have*

$$S_N(H_{\mathbf{k}}, \mathcal{P}) = \begin{cases} b^{-rn} & \textit{for} \quad (b-1)^r b^{rn} \quad \textit{vectors} \quad \mathbf{k} \in \Delta(\mathbf{g}) \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. **(1)** W. l. o. g., we consider the case

$$0 \le g_1, \dots, g_l < n \quad \text{and} \quad g_{l+1}, \dots, g_r \ge n, \quad 1 \le l \le r.$$

The fundamental domain $D_{\mathbf{k}}$ of $h_{\mathbf{k}}$, $\mathbf{k} \in \Delta(\mathbf{g})$ has the form

$$D_{\mathbf{k}} = \prod_{i=1}^{r} \left[\frac{k_i(g_i)}{b^{g_i}}, \frac{k_i(g_i)+1}{b^{g_i}} \right[\times [0,1[^{s-r} =: \prod_{i=1}^{s} I_i. \tag{4.3}$$

Two cases are possible:
Case I: $D_{\mathbf{k}} \cap \mathcal{P} = \emptyset$, then $S_N(h_{\mathbf{k}}, \mathcal{P}) = 0$.
Case II:

$$\sharp(D_{\mathbf{k}} \cap \mathcal{P}) = b^{(s-r)n} \prod_{i=1}^{l} b^{n-g_i}\,;$$

In the second case, each of the intervals I_i, $1 \le i \le l$, has b^{n-g_i} possibilities for the i-th coordinate, and the intervals I_i, $l+1 \le i \le r$, contain exactly one coordinate x_i of a lattice point $\mathbf{x} \in \mathcal{P}$. Further, $H_{k_i}(x_i) = 1$ as the coordinate x_i necessarily belongs to $D_{k_i}(0)$. Finally, each of the I_i, $r+1 \le i \le s$, has b^n possibilities for the i-th coordinate $x_i^{(j)}$, $0 \le j \le b^n - 1$, where $H_0(x_i^{(j)}) = 1$.

The intervals $I_1, \dots, I_l$ are partitioned into b elementary b-adic subintervals

$$D_{k_i}(a_i) = \left[\frac{b \cdot k_i(g_i) + a_i}{b^{g_i+1}}, \frac{b \cdot k_i(g_i) + a_i + 1}{b^{g_i+1}} \right[, \quad 0 \le a_i \le b-1, \quad 1 \le i \le l.$$

Each of these subintervals contains $b^{n-g_i-1} \ge 1$ coordinates. Hence we get

$$S_N(H_{\mathbf{k}}, \mathcal{P}) = \frac{1}{N} b^{(s-r)n} \prod_{j=1}^{l} b^{n-g_j-1} \prod_{i=1}^{l} \underbrace{\sum_{a_i=0}^{b-1} e_b(a_i \cdot (k_i)_{g_i})}_{=0}.$$

(2) Similar to part (1), if $D_{\mathbf{k}} \cap \mathcal{P} = \emptyset$, then $S_N(h_{\mathbf{k}}, \mathcal{P}) = 0$. Otherwise, we have, $\sharp(D_{\mathbf{k}} \cap \mathcal{P}) = b^{n(s-r)}$. Since, from $\lambda(I_i) \le b^{-n}$, $1 \le i \le r$, it follows that the first r coordinates $(\frac{\bar{l}_1}{b^n}, \dots, \frac{\bar{l}_r}{b^n})$ of a point $\mathbf{p} \in D_{\mathbf{k}} \cap \mathcal{P}$ are fixed. The remaining coordinates vary in $\{0, 1/b^n, \dots, (b^n-1)/b^n\}$. Therefore

$$\begin{aligned} S_N(H_{\mathbf{k}}, \mathcal{P}) &= \frac{1}{N} H_{k_1}(\tfrac{\bar{l}_1}{b^n}) \cdots H_{k_r}(\tfrac{\bar{l}_r}{b^n}) \sum_{l_{r+1},\dots,l_s=0}^{b^n-1} H_0(\tfrac{l_{r+1}}{b^n}) \cdots H_0(\tfrac{l_s}{b^n}) \\ &= \frac{1}{b^{nr}} \prod_{i=1}^{r} H_{k_i}(\tfrac{\bar{l}_i}{b^n}). \end{aligned}$$

For any number $x = \bar{l}_i/b^n$, we obtain that the digit $x_{g_i} = 0$, because of $g_i \ge n$, $1 \le i \le r$. From this, it follows that $H_{k_i}(x) = e_b(x_{g_i} \cdot (k_i)_{g_i}) = 1$.

There are b^{nr} possible $D_{\mathbf{k}}$ with $\sharp(D_{\mathbf{k}} \cap \mathcal{P}) = b^{n(s-r)}$. For a given $D_{\mathbf{k}}$, there are $(b-1)^r$ different $\mathbf{k} \in \Delta(\mathbf{g})$ to get this particular fundamental domain. We only have to vary the digit $(k_i)_{g_i} \in \{1,\ldots,b-1\}$ for $1 \le i \le r$. Thus we get $S_N(H_{\mathbf{k}},\mathcal{P}) = b^{-nr}$ for $(b-1)^r b^{nr}$ points $\mathbf{k} \in \Delta(\mathbf{g})$, and $S_N(H_{\mathbf{k}},\mathcal{P}) = 0$ otherwise. □

We continue with Part (a) of the proof of Theorem 3.1. For a given $\mathbf{g} \in \mathbf{N}_1^s$ with property (4.2), let

$$S_{\mathbf{g}} := \sum_{\mathbf{k}\in\Delta(\mathbf{g})} \frac{1}{r_b(\mathbf{k})^\alpha} \cdot |S_N(h_{\mathbf{k}},\mathcal{P})|.$$

Using Lemma 4.1, we only have to consider $\mathbf{g}$ with $g_1,\ldots,g_r \ge n$. In this case we obtain

$$S_{\mathbf{g}} \le (b-1)^r b^{(\frac{1}{2}-\alpha)(g_1+\ldots+g_r)}.$$

By partitioning the sum on the right side of (4.1) into areas $\Delta(\mathbf{g})$, we get

$$R_N(f,\mathcal{P}) \le C\sum_{r=1}^{s} \binom{s}{r}(b-1)^r \sum_{i=rn}^{\infty} \sum_{\substack{g_1,\ldots,g_r=n \\ g_1+\ldots+g_r=i}}^{\infty} b^{(\frac{1}{2}-\alpha)i}.$$

There are $\binom{i-rn+r-1}{r-1} \le (i-rn+1)^{r-1}$ solutions of $g_1+\ldots+g_r = i$ with $g_j \ge n$, $1 \le j \le r$ (see [1, Sect. 1.1]). This yields

$$\begin{aligned} R_N(f,\mathcal{P}) &\le C\sum_{r=1}^{s} \binom{s}{r}(b-1)^r b^{(\frac{1}{2}-\alpha)(rn-1)} \underbrace{\sum_{i=1}^{\infty} i^{s-1} b^{(\frac{1}{2}-\alpha)i}}_{=:A} \\ &\le C \cdot A \cdot b^{(\alpha-\frac{1}{2})} b^s \cdot N^{-\frac{\alpha-1/2}{s}}. \end{aligned}$$

Part (b) We consider the function $f : [0,1[^s \longrightarrow \mathbf{R}$,

$$f(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbf{N}_0^s} \hat{f}(\mathbf{k}) \cdot h_{\mathbf{k}}(\mathbf{x}),$$

with $\hat{f}(k_1,0,\ldots,0) = b^{-g\alpha}$ for $k_1 \in \Delta(g)$, $g \ge 0$, and $\hat{f}(\mathbf{k}) = 0$ otherwise. This definition yields $f \in {}_b\tilde{E}_s^\alpha(1)$, and with $\mathbf{g} := (g_1,-1,\ldots,-1)$, $g_1 \ge 0$,

$$R_N(f,\mathcal{P}) = \left| \sum_{g_1=0}^{\infty} \sum_{\mathbf{k}\in\Delta(\mathbf{g})} \frac{1}{b^{g_1\alpha}} S_N(h_{\mathbf{k}},\mathcal{P}) \right|.$$

Using $S_N(h_{\mathbf{k}},\mathcal{P}) = b^{\frac{g_1}{2}} S_N(H_{\mathbf{k}},\mathcal{P})$ and Lemma 4.1 for $r=1$, we get

$$R_N(f,\mathcal{P}) = \sum_{g_1=n}^{\infty} \frac{b-1}{b^{g_1(\alpha-\frac{1}{2})}} = \frac{(b-1)b^{(\alpha-\frac{1}{2})}}{b^{(\alpha-\frac{1}{2})}-1} \cdot \frac{1}{N^{\frac{\alpha-1/2}{s}}}.$$

Part (c) Let $\mathcal{P}$ be a (t,m,s)-net in base b and $\mathbf{g} \in \mathbf{N}_1^s$ with property (4.2).

LEMMA 4.2.
(1) *$S_N(h_{\mathbf{k}},\mathcal{P})=0$ for all $\mathbf{k}\in\Delta(\mathbf{g})$ and $\mathbf{g}$ with $\sum_{i=1}^s(g_i+1)\le m-t$.*
(2) *$|S_N(H_{\mathbf{k}},\mathcal{P})|\le b^{-(g_1+\ldots+g_r)}$ for all $\mathbf{k}\in\Delta(\mathbf{g})$, and $\mathbf{g}$ with*

$$m-t<\sum_{i=1}^s(g_i+1)<m-t+r.$$

Note that this case is only possible for $2\le r\le s$.
(3) *If $\sum_{i=1}^s(g_i+1)\ge m-t+r$, then there are at most $(b-1)^r b^m$ vectors $\mathbf{k}\in\Delta(\mathbf{g})$ with $|S_N(H_{\mathbf{k}},\mathcal{P})|\ne 0$. In this case we have $|S_N(H_{\mathbf{k}},\mathcal{P})|\le b^{t-m}$.*

PROOF. **(1)** The fundamental domain $D_{\mathbf{k}}$ of $h_{\mathbf{k}}$, $\mathbf{k}\in\Delta(\mathbf{g})$, is partitioned into b^r elementary b-adic subintervals of the form

$$I=\prod_{i=1}^r D_{k_i}(a_i)\times[0,1[^{s-r},$$

where the a_i vary in $\{0,\ldots,b-1\}$. The function $h_{\mathbf{k}}$ is obviously constant on each interval I. The requirement $\sum_{i=1}^s(g_i+1)\le m-t$ yields $\lambda_s(I)\ge b^{t-m}$ so that each I contains exactly $\lambda_s(I)\cdot b^m$ points of $\mathcal{P}$. Hence, we have

$$S_N(h_{\mathbf{k}},\mathcal{P})=\frac{1}{b^{(g_1+\ldots+g_r)+r}}\sum_{a_1=0}^{b-1}\cdots\sum_{a_r=1}^{b-1}\prod_{i=1}^r b^{\frac{g_i}{2}}e_b(a_i\cdot k_{g_i})\;=\;0.$$

(2) The inequality $m-t-r<g_1+\ldots+g_r<m-t$ yields $b^{t-m}<\lambda(D_{\mathbf{k}})<b^{r+t-m}$. Therefore $\sharp(D_{\mathbf{k}}\cap\mathcal{P})=b^m\lambda(D_{\mathbf{k}})$ and thus we get $|S_N(H_{\mathbf{k}},\mathcal{P})|\le\lambda(D_{\mathbf{k}})=b^{-(g_1+\ldots+g_r)}$.
(3) In this case we have $\lambda(D_{\mathbf{k}})\le b^{t-m}$. Hence, $D_{\mathbf{k}}$ contains at most b^t elements of $\mathcal{P}$. The result follows easily from Remark 2.3. □

We continue with the proof of Part (c) of Theorem 3.1 . Again, we consider

$$S_{\mathbf{g}}:=\sum_{\mathbf{k}\in\Delta(\mathbf{g})}\frac{1}{r_b(\mathbf{k})^\alpha}\cdot|S_N(h_{\mathbf{k}},\mathcal{P})|.$$

For the case (2) of the lemma above, we get $S_{\mathbf{g}}\le(b-1)^r\cdot b^{(\frac12-\alpha)(g_1+\ldots+g_r)}$, and for the case (3) $S_{\mathbf{g}}\le(b-1)^r\cdot b^t\cdot b^{(\frac12-\alpha)(g_1+\ldots+g_r)}$. This and part (1) of the lemma above yield

$$\begin{aligned}R_N(f,\mathcal{P})\quad&\le\quad C\sum_{r=2}^s\binom{s}{r}(b-1)^r\sum_{\substack{g_1,\ldots,g_r=0\\ m-t-r<g_1+\ldots+g_r<m-t}}^{m-t-1}b^{(\frac12-\alpha)(g_1+\ldots+g_r)}\\&\quad+C\cdot b^t\sum_{r=1}^s\binom{s}{r}(b-1)^r\sum_{\substack{g_1,\ldots,g_r=0\\ g_1+\ldots+g_r\ge m-t}}^{\infty}b^{(\frac12-\alpha)(g_1+\ldots+g_r)}.\end{aligned}$$

The number of solutions of the equation $g_1 + \ldots + g_r = i,\ g_i \in \mathbf{N}_0$, equals $\binom{i+r-1}{r-1} \leq (i+1)^{r-1}$. This yields

$$\begin{aligned} R_N(f,\mathcal{P}) \quad &\leq \quad C \sum_{r=2}^{s} \binom{s}{r} (b-1)^r \sum_{i=1}^{r-1} (m-t-r+1+i)^{s-1} b^{(\frac{1}{2}-\alpha)(m-t-r+i)} \\ &\quad + C b^t \sum_{r=1}^{s} \binom{s}{r} (b-1)^r \sum_{i=r}^{\infty} (m-t-r+1+i)^{s-1} b^{(\frac{1}{2}-\alpha)(m-t-r+i)}. \end{aligned}$$

For all $i \in \mathbf{N}$ we have $m-t-r+1+i \leq m \cdot i$ for $r \geq 2$, and $m-t-r+1+i \leq (m+1)\cdot i$ for $r \geq 1$. Hence

$$\begin{aligned} R_N(f,\mathcal{P}) \quad &\leq \quad C \cdot \frac{m^{s-1} b^{(\alpha-\frac{1}{2})(t+s)}}{b^{(\alpha-\frac{1}{2})m}} \sum_{r=2}^{s} \binom{s}{r} (b-1)^r \sum_{i=1}^{\infty} i^{s-1} b^{(\frac{1}{2}-\alpha)i} \\ &\quad + C \cdot \frac{m^{s-1} 2^{s-1} b^t b^{(\alpha-\frac{1}{2})(t+s)}}{b^{(\alpha-\frac{1}{2})m}} \sum_{r=1}^{s} \binom{s}{r} (b-1)^r \sum_{i=1}^{\infty} i^{s-1} b^{(\frac{1}{2}-\alpha)i}. \end{aligned}$$

Again let $A := \sum_{i=1}^{\infty} i^{s-1} b^{(\frac{1}{2}-\alpha)i}$, then we get the result

$$R_N(f,\mathcal{P}) \leq \frac{C \cdot A \cdot (1+2^{s-1}) b^{(\alpha+\frac{1}{2})(t+s)}}{(\log b)^{s-1}} \cdot \frac{(\log N)^{s-1}}{N^{\alpha-\frac{1}{2}}}.$$

Part (d) is proven in Entacher[2].

Proof of Theorem 3.2

Let $B = b^L$, $L \geq 2$, for a given base $b \geq 2$. We shall prove the result by induction for dimension s.

Case I: dimension $s = 1$

We start with the examination of the Haar series $S_{h_k}^{(B)}$, $h_k \in \mathcal{H}_b$, with respect to the system $\mathcal{H}_B$. To avoid too many indices, we shall denote the Haar coefficients of a given function f with respect to the system $\mathcal{H}_B$ by $\hat{f}(n)$, hence with argument n. Further we will use the argument k to signify that $\hat{f}(k)$ denotes the Haar coefficient of f with respect to the Haar system $\mathcal{H}_b$. In the next lemma, we shall prove that the k-th Haar function $h_k \in \mathcal{H}_b$ is a Haar polynomial with respect to the system $\mathcal{H}_B$.

LEMMA 4.3. *Let $k \in \Delta_B(j)$, $j \geq 0$. Then the n-th Haar coefficient $\widehat{h_k}(n)$ with respect to the system $\mathcal{H}_B$ of the k-th Haar function $h_k \in \mathcal{H}_b$ is equal to zero for all n with $n < B^j$ or $n \geq B^{j+1}$. If $k = 0$, then $\widehat{h_k}(n) = 0$ for all $n \geq 1$.*

PROOF. We have

$$\widehat{h_k}(n) = \int_{D_k \cap D_n^{(B)}} h_k(x) \overline{h_n^{(B)}}(x)\, dx.$$

The case $k = 0$ is easily verified. Let $j \geq 0$ and $n \geq B^{j+1}$. Then we have either $D_k \cap D_n^{(B)} = \emptyset$ or $D_n^{(B)} \subset D_k$. In the second case, the Haar function h_k, is constant on $D_k \cap D_n^{(B)} = D_n^{(B)}$ and therefore $\widehat{h_k}(n) = 0$. The inequality $n < B^j$ is treated similarly. □

COROLLARY 4.4. *Let $B^j \leq k < B^{j+1}$, $j \in \mathbf{N}_0$. The Haar function $h_k \in \mathcal{H}_b$ has a finite Haar series with respect to the system $\mathcal{H}_B$, and*

$$h_k(x) = \sum_{n \in \Delta_B(j)} \widehat{h_k}(n) h_n^{(B)}(x).$$

Let $f \in {}_b\tilde{E}_1^\alpha(C)$. Our goal is to estimate the Haar coefficients $\hat{f}(n)$, $n \in \mathbf{N}$. The following lemma shows a representation of the numbers $\hat{f}(n)$ in terms of the Haar coefficients $\hat{f}(k)$ with regard to $\mathcal{H}_b$ and the Haar coefficients $\widehat{h_k}(n)$ with respect to the function system $\mathcal{H}_B$.

LEMMA 4.5. *If $f \in {}_b\tilde{E}_1^\alpha(C)$, $\alpha > 1/2$, then*

$$\hat{f}(n) = \sum_{g=L\bar{g}}^{L(\bar{g}+1)-1} \sum_{k \in \Delta(g)} \hat{f}(k)\widehat{h_k}(n) \quad \textit{for} \quad n \in \Delta_B(\bar{g}), \quad \bar{g} \geq 0.$$

This lemma will yield the order of $\hat{f}(n)$ if we are able to estimate $\widehat{h_k}(n)$.

PROOF. We have

$$S_f(x) = \hat{f}(0) + \sum_{\bar{g}=0}^{\infty} \sum_{g=\bar{g}L}^{(\bar{g}+1)L-1} \sum_{k \in \Delta(g)} \hat{f}(k) h_k(x).$$

Since $k \in \Delta(g)$ with $\bar{g}L \leq g < (\bar{g}+1)L$ if and only if $B^{\bar{g}} \leq k < B^{\bar{g}+1}$, Corollary 4.4 yields

$$\begin{aligned} S_f(x) &= \hat{f}(0) + \sum_{\bar{g}=0}^{\infty} \sum_{g=\bar{g}L}^{(\bar{g}+1)L-1} \sum_{k \in \Delta(g)} \hat{f}(k) \sum_{n \in \Delta_B(\bar{g})} \widehat{h_k}(n) h_n^{(B)}(x) \\ &= \hat{f}(0) + \sum_{\bar{g}=0}^{\infty} \sum_{n \in \Delta_B(\bar{g})} \left(\sum_{g=\bar{g}L}^{(\bar{g}+1)L-1} \sum_{k \in \Delta(g)} \hat{f}(k)\widehat{h_k}(n) \right) h_n^{(B)}(x). \end{aligned}$$

This yields the result. □

Let us consider the integers k, n, with $B^{\bar{g}} \leq k, n < B^{\bar{g}+1}$, $\bar{g} \geq 0$, since from Lemma 4.3 it follows that only in these cases $\widehat{h_k}(n) \neq 0$ is possible.

In the following, let $n \in \Delta_B(\bar{g})$, $\bar{g} \geq 0$, and $k \in \Delta(g)$ with $L\bar{g} \leq g < L(\bar{g}+1)$, more precisely

$$\text{(4.4)} \qquad n \in \Delta_B(\bar{g}) \quad \text{and} \quad k \in \Delta(g) \quad \text{with} \quad g = L\bar{g} + j, \quad 0 \leq j < L.$$

Further let us define

$$E_a := D_n^{(B)}(a) = [\beta^{(a)}, \beta^{(a+1)}[\quad \text{with} \quad \beta^{(a)} := \frac{n(\bar{g})}{B^{\bar{g}}} + \frac{a}{B^{\bar{g}+1}}, \quad 0 \leq a < B.$$

Since

$$h_n^{(B)}(x) = B^{\frac{\bar{g}}{2}} \sum_{a=0}^{B-1} e_B(a \cdot n_{\bar{g}}) \cdot \mathbf{1}_{E_a}(x),$$

we get

$$\overline{\widehat{h_k}(n)} = B^{\frac{\bar{g}}{2}} \sum_{a=0}^{B-1} e_B(a \cdot n_{\bar{g}}) \cdot \widehat{\mathbf{1}_{E_a}}(k). \tag{4.5}$$

From Hellekalek [3, Lemma 3.2], we conclude that we may have $\widehat{\mathbf{1}_{E_a}}(k) \neq 0$ only if $k(g) \in \{b^g \cdot \beta^{(a)}(g), b^g \cdot \beta^{(a+1)}(g)\}$, and in this case the given estimation of $|\widehat{\mathbf{1}_{E_a}}(k)|$ does not depend on the explicit value of $k(g)$. Therefore we have to examine the set

$$\mathcal{A}(k,g) := \{a \in \{0,\ldots,B-1\} \;:\; k(g) = b^g \cdot \beta^{(a)}(g)\}.$$

The case $k(g) = b^g \beta^{(B)}(g)$ will be treated below.

Let $n(\bar{g}) = \bar{n}_0 + \bar{n}_1 B + \ldots + \bar{n}_{\bar{g}-1} B^{\bar{g}-1}$ be the B-adic expansion of $n(\bar{g})$. Changing to base b yields $n(\bar{g}) = n_0 + n_1 b + \ldots + n_{L\bar{g}-1} b^{L\bar{g}-1}$, $n_i \in \{0,\ldots,b-1\}$, and from this we get for $a = a_0 + a_1 b + \ldots + a_{L-1} b^{L-1}$,

$$n(\bar{g}) \cdot B + a = a_0 + a_1 b + \ldots + a_{L-1} b^{L-1} + n_0 b^L + \ldots + n_{L\bar{g}-1} b^{L(\bar{g}+1)-1}.$$

Because of $g = L\bar{g} + j$, $j \in \{0,\ldots,L-1\}$, it follows

$$\beta^{(a)}(g) = \begin{cases} 0.n_{L\bar{g}-1} n_{L\bar{g}-2} \ldots n_0 a_{L-1} \ldots a_{L-j} & \text{if } \; j \geq 1 \\ 0.n_{L\bar{g}-1} \ldots n_0 & \text{if } \; j = 0, \end{cases}$$

and $\beta_g^{(a)} = a_{L-j-1}$. Thus we have

$$b^g \cdot \beta^{(a)}(g) = a_{L-j} + a_{L-j+1} b + \ldots + a_{L-1} b^{j-1} + n_0 b^j + \ldots + n_{L\bar{g}-1} b^{g-1}.$$

The requirement $k(g) = b^g \cdot \beta^{(a)}(g)$ is equivalent to $k_0 = a_{L-j}, \ldots, k_{j-1} = a_{L-1}, k_j = n_0, \ldots, k_{g-1} = n_{L\bar{g}-1}$. Hence, the numbers a have the form

$$a = a_0 + a_1 b + \ldots + a_{L-j-1} b^{L-j-1} + k_0 b^{L-j} + \ldots + k_{j-1} b^{L-1}.$$

If we vary $a_0, \ldots, a_{L-j-1}$, we get

$$\mathcal{A}(g,k) = \{b^{L-j} \cdot k(j) + i \;:\; i \in \{0,\ldots,b^{L-j}-1\}\} \quad \text{and} \quad \sharp\mathcal{A}(g,k) = b^{L-j}.$$

Note that if $a = b^{L-j} \cdot k(j)$, then $a_0 = 0, \ldots, a_{L-j-1} = 0$, and therefore

$$\beta_g^{(a)} = 0 \quad \text{and} \quad \beta^{(a)} - \beta^{(a)}(g+1) = 0. \tag{4.6}$$

Finally, we have to consider the case $\beta^{(B)} = (n(\bar{g})+1)/B^{\bar{g}}$, since the case $k(g) = b^g\beta^{(B)}(g)$ is not included above. Here we distinguish between $\beta^{(B)} = 1$ and $\beta^{(B)} < 1$. Let $a := B-1$. The first case yields $\widehat{\mathbf{1}_{E_a}}(k) = -\widehat{\mathbf{1}_J}(k)$, $J := [0, \beta^{(a)}[$, and therefore we have $\widehat{\mathbf{1}_{E_a}}(k) \neq 0$ only if $a \in \mathcal{A}(g,k)$ due to the above proof. If $\beta^{(B)} < 1$, one can easily show that $\beta_g^{(B)} = 0$ and $\beta^{(B)} - \beta^{(B)}(g+1) = 0$. From Lemma 3.1 in [3], we observe that $\widehat{\mathbf{1}_{E_a}}(k) = -\widehat{\mathbf{1}_J}$, $J := [0, \beta^{(a)}[$, and we are in the same situation as before.
Let us return to the examination of $\widehat{h_k}(n)$ in (4.5).

LEMMA 4.6.
If $n \in \Delta_B(\bar{g})$, $\bar{g} \geq 0$, and $k \in \Delta(g)$ with $g = L\bar{g}+j$, $0 \leq j < L$, then there are exactly $(b-1)\cdot b^j$ integers[1] $k \in \Delta(g)$ with $|\widehat{h_k}(n)| \neq 0$. In this case we have

$$|\widehat{h_k}(n)| \leq B^{\frac{\bar{g}}{2}} \cdot b^{L-j} \cdot b^{-\frac{g}{2}-1} \frac{1}{\sin \pi \frac{k_g}{b}}.$$

PROOF. The considerations above yield

$$\mathcal{A}(g,k) = \{b^{L-j}k(j) + i \ : \ i \in \{0, \ldots, b^{L-j}-1\}\}.$$

Let $k(j) \neq 0$. We consider $a = b^{L-j}\cdot k(j) - 1$. Then $a \notin \mathcal{A}(g,k)$, but $a+1 = b^{L-j}\cdot k(j) \in \mathcal{A}(g,k)$. In this case, (4.6) implies that $\beta_g^{(a+1)} = 0$ and $\beta^{(a+1)} - \beta^{(a+1)}(g+1) = 0$. Hence Lemma 3.1, Part 2 of Hellekalek [3] implies $\widehat{\mathbf{1}_{E_a}}(k) = 0$. If $k(j) = 0$, the situation above is not possible, since $\mathcal{A}(g,k) = \{0, \ldots, b^{L-j}-1\}$. Hence for both cases, $k(j) \neq 0$ and $k(j) = 0$, we obtain, by equation (4.5),

$$\overline{\widehat{h_k}(n)} = B^{\frac{\bar{g}}{2}} \cdot \sum_{i=0}^{b^{L-j}-1} e_B(a_i \cdot n_{\bar{g}}) \cdot \widehat{\mathbf{1}_{E_{a_i}}}(k) \quad \text{with} \quad a_i := b^{L-j}k(j) + i.$$

The result follows from Lemma 3.2 in [3] and from the fact that there are b^j different possibilities for $k(j)$ and $b-1$ possibilities for k_g. □

Now we are able to estimate $\hat{f}(n)$. Lemma 4.5 yields

$$|\hat{f}(n)| \leq C \sum_{g=L\bar{g}}^{L(\bar{g}+1)-1} \sum_{k\in\Delta(g)} \frac{1}{b^{g\alpha}} \cdot |\widehat{h_k}(n)|.$$

Using $j = g - L\bar{g}$ and the lemma above, we get

$$\sum_{k\in\Delta(g)} \frac{1}{b^{g\alpha}} \cdot |\widehat{h_k}(n)| \leq \frac{1}{b^{g(\alpha+\frac{1}{2})}} \cdot B^{(\frac{\bar{g}}{2}+1)} \cdot \underbrace{\frac{1}{b}\sum_{a=1}^{b-1} \frac{1}{\sin \pi \frac{a}{b}}}_{=:\,C(b)}.$$

[1]These are exactly the numbers k with $k_g \in \{1, \ldots, b-1\}$, and $k(g)$ varies arbitrarily in $\{0, 1, \ldots, b^j - 1\}$.

Finally we observe that

$$|\hat{f}(n)| < \underbrace{C \cdot C(b) \cdot B \cdot \frac{b^{(\alpha+\frac{1}{2})}}{b^{(\alpha+\frac{1}{2})}-1}}_{=: C(\alpha,b)} \cdot \frac{1}{B^{\alpha \cdot \bar{g}}}.$$

Case II: dimension $s \geq 2$

The proof of this case is obtained by induction, in same way as in [7, p. 709]. In the latter paper a similar theorem is proved for the Walsh case.

Acknowledgement.

I want to thank my supervisor Peter Hellekalek for his support during the work on my PhD thesis. Further, I want to thank Wolfgang Ch. Schmid for his assistance with computer programs and numerical results.

REFERENCES

1. K. Entacher. *Generalized Haar function systems in the theory of uniform distributions of sequences modulo one.* PhD thesis, University of Salzburg, 1995.
2. K. Entacher. Generalized Haar function systems, digital nets and quasi-Monte Carlo integration. In H.H. Szu, editor, *Wavelet Applications III, Proc. SPIE 2762*, 1996. Available on the internet at `http://random.mat.sbg.ac.at`.
3. P. Hellekalek. General discrepancy estimates III: The Erdös-Turàn-Koksma Inequality for the Haar function system. *Mh. Math.*, **120**:25–45, 1995.
4. N.M. Korobov. *Number-Theoretic Methods in Approximate Analysis.* Fizmatgiz, Moscow, 1963. (In Russian).
5. G. Larcher, A. Lauß, H. Niederreiter, and W.Ch. Schmid. Optimal polynomials for (t, m, s)-nets and numerical integration of Walsh series. *SIAM J. Numer. Analysis*, **33**, 1996 pp. 2239–2253.
6. G. Larcher and W.Ch. Schmid. Multivariate Walsh series, digital nets and quasi-Monte Carlo integration. In H. Niederreiter and P. Jau-Shyong Shiue, editors, *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, volume 106 of *Lecture Notes in Statistics*, pages 252–262. Springer, 1995.
7. G. Larcher, W.Ch. Schmid, and R. Wolf. Representation of functions as Walsh series to different bases and an application to the numerical integration of high-dimensional Walsh series. *Math. Comp.*, **63**:701–716, 1994.
8. G. Larcher, W.Ch. Schmid, and R. Wolf. Quasi-Monte Carlo methods for the numerical integration of multivariate Walsh series. *Mathl. Comput. Modelling*, **23**(8/9):55–67, 1996.

9. G. Larcher and C. Traunfellner. On the numerical integration of Walsh series by number-theoretic methods. *Math. Comp.*, **63**:277–291, 1994.

10. H. Niederreiter. Point sets and sequences with small discrepancy. *Mh. Math.*, **104**:273–337, 1987.

11. H. Niederreiter. Low-discrepancy and low-dispersion sequences. *J. Number Theory*, **30**:51–70, 1988.

12. H. Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods.* SIAM, Philadelphia, USA, 1992.

13. I. M. Sobol'. The distribution of points in a cube and the approximate evaluation of integrals. *Zh. Vycisl. Mat. i Mat Fiz.*, **7**:784–802, 1967. (In Russian).

14. I. M. Sobol'. *Multidimensional Quadrature Formulas and Haar Functions.* Nauka., Moscow, 1969. (In Russian).