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Discrepancy estimates based on Haar functions

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Abstract

We present a technique to estimate the star-discrepancy of (t, m, s)-nets using generalized Haar function systems and apply this technique to obtain upper bounds for the star-discrepancy of special digital (t, m, s)-nets in base 2 and dimension $s = 2$. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Niederreiter [12,14] has developed different methods to estimate the star-discrepancy of (t, m, s)-nets in base b. The proofs use the general definition of these nets and combinatorial methods.

In this paper we present a technique to estimate the star-discrepancy of (t, m, s)-nets using generalized Haar function systems. The basis of this technique is the inequality of Erdös-Turán-Koksma for the Haar function system shown in Hellekalek [7]. The 'raw form' of this inequality ([7], Theorem 3.4) provides an upper bound of the (star) discrepancy of a point set $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{N-1}$ in the s-dimensional unit cube $[0, 1[^s$, $s \geq 1$, in terms of Weyl sums for Haar systems. Estimation of these Weyl sums provides an interesting method to determine bounds of the star-discrepancy of (t, m, s)-nets.

We apply the latter technique to obtain improved upper bounds for the star-discrepancy of special digital (t, m, s)-nets in base 2 and in dimension $s = 2$. Our method will further lead us to the exact determination of the discrete discrepancy which is the natural measure of equidistribution for point sets with finite precision such as digital (t, m, s)-nets. Advantages and drawbacks of the technique are discussed.

Definition 1. Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{N-1}$ be a point set in $[0, 1[^s$. The star-discrepancy $D_N^*(\mathcal{P})$ of $\mathcal{P}$ is defined as

$$D_N^*(\mathcal{P}) := \sup_{J \in \mathcal{J}^*} \left| \frac{1}{N} \cdot \#\{n : \boldsymbol{x}_n \in J, 0 \leq n < N\} - \lambda_s(J) \right|, \tag{1}$$

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where $\mathcal{J}^*$ denotes the class of all subintervals J of $[0, 1[^s$ of the form $J = \Pi_{i=1}^{s}[0, u_i[, 0 \leq u_i \leq 1$, $1 \leq i \leq s$, $\#\mathcal{M}$ denotes the number of elements of a set $\mathcal{M}$ and λ_s the Lebesgue measure on $[0, 1[^s$.

If we define $f_J(\boldsymbol{x}) := \mathbf{1}_J(\boldsymbol{x}) - \lambda_s(J), \boldsymbol{x} \in [0, 1[^s$, with the characteristic function $\mathbf{1}_J$, we get

$$D_N^*(\mathcal{P}) = \sup_{J \in \mathcal{J}^*} |R_N(f_J, \mathcal{P})| \quad \text{where } R_N(f_J, \mathcal{P}) := \frac{1}{N}\sum_{n=0}^{N-1} f_J(\boldsymbol{x}_n). \tag{2}$$

Note that $R_N(f_J, \mathcal{P})$ equals the Monte Carlo approximation of $\int_{[0,1[^s} f_J \, d\lambda_s$.

The concept of (*t*, *m*, *s*)-nets in base 2 was introduced by Sobol' [17]. Niederreiter has extended this idea considerably. He has introduced arbitrary bases, efficient construction methods for such point sets and a comprehensive theory, see the monograph [14] and the recent survey [16].

Definition 2. Let $0 \leq t \leq m$ and $b \geq 2$ be integers. A (*t*, *m*, *s*)-net in base *b* is a point set $\mathcal{P}$ consisting of b^m points in $[0, 1[^s$ such that every elementary interval I in base *b* with volume $\lambda_s(I) = b^{t-m}$ contains exactly b^t points of $\mathcal{P}$. Note that such an elementary interval I in base *b*, by definition, equals $I = \prod_{i=1}^{s}[a_i/b^{g_i}, (a_i + 1)/b^{g_i}[$, where $a_i, g_i \in \boldsymbol{Z}$, $g_i \geq 0$ and $0 \leq a_i \leq b^{g_i}$ for $1 \leq i \leq s$.

2. Generalized Haar function systems

In the following we fix an arbitrary integer base $b \geq 2$. For the notations and definitions of generalized Haar functions relative to base *b* and further applications, see [1,3,6,7]. In this section we recall the basic notations.

For an integer $k \geq 0$ and an arbitrary number $x \in [0, 1[$, let $k = \sum_{j=0}^{\infty} k_j b^j$ and $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, k_j, $x_j \in \{0, 1, \ldots, b-1\}$, be the *b*-adic expansions of *k* and *x* in base *b*. For $g \in \boldsymbol{N}$ we define $k(g) := \sum_{j=0}^{g-1} k_j b^j$ and $x(g) := \sum_{j=0}^{g-1} x_j b^{-j-1}$. Further let $k(0) := 0$ and $x(0) := 0$. The support of a given Haar function h_k, $k \geq 0$ is equal to an elementary *b*-adic interval. We now define sets of integers *k*, for which such intervals have the same length (resolution).

Definition 3.

1. Let *g* be a nonnegative integer. Then $\Delta(g) := \{k \in \boldsymbol{N} : b^g \leq k < b^{g+1}\}$. Further, let $\Delta(-1) := \{0\}$, and the sets $\boldsymbol{N}_0 := \boldsymbol{N} \cup \{0\}$, and $\boldsymbol{N}_1 := \boldsymbol{N}_0 \cup \{-1\}$.
2. If $\boldsymbol{g} = (g_1, \ldots, g_s)$, $s \geq 2$ and $g_i \in \boldsymbol{N}_1$, then $\Delta(\boldsymbol{g}) := \prod_{i=1}^{s} \Delta(g_i)$.

Definition 4. Let $e_b : \boldsymbol{Z}_b \to \boldsymbol{K}$, where $\boldsymbol{Z}_b = \{0, \ldots, b-1\}$ is the least residue system modulo *b*, and $K := \{z \in \boldsymbol{C} : |z| = 1\}$, denote the function $e_b(a) := \exp(2\pi i a/b)$, $(a \in \boldsymbol{Z}_b)$. The *k*-th Haar function h_k, $k \geq 0$, to the base *b* is defined as follows: If $k = 0$ $(g = -1)$, then $h_0(x) := 1, \forall x \in [0, 1[$. If $k \in \Delta(g)$, $g \geq 0$, then

$$h_k(x) := b^{g/2} \sum_{a=0}^{b-1} e_b(a k_g) \cdot \mathbf{1}_{D_k(a)}(x), \tag{3}$$

with elementary b-adic intervals $D_k(a) := [(bk(g)+a)/b^{g+1}, (bk(g)+a+1)/b^{g+1}[$. The k-th normalized Haar function H_k on $[0, 1[$ is defined as $H_0 := h_0$ and, if $k \in \Delta(g)$, $g \geq 0$, then $H_k := b^{-g/2} h_k$.

Hence, the support D_k of the k-th Haar function h_k is given as the following elementary b-adic interval: If $k = 0$, then $D_0 := [0, 1[$. If $k \in \Delta(g)$, $g \geq 0$, then $D_k := \cup_{a=0}^{b-1} D_k(a) = [k(g)/b^g, (k(g)+1)/b^g[$.

Definition 5. Let $\mathcal{H}_b := \{h_{\boldsymbol{k}} : \boldsymbol{k} := (k_1, \ldots, k_s) \in \boldsymbol{N}_0^s\}$ denote the Haar function system to the base b on the s-dimensional torus $[0, 1[^s$, $s \geq 1$. The $\boldsymbol{k}$-th Haar function $h_{\boldsymbol{k}}$ is defined as $h_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{i=1}^{s} h_{k_i}(x_i)$, $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0, 1[^s$. The normalized version $H_{\boldsymbol{k}}$ is defined in the same way and the supports of $h_{\boldsymbol{k}}$ and $H_{\boldsymbol{k}}$ are defined as $D_{\boldsymbol{k}} := \prod_{i=1}^{s} D_{k_i}$.

3. The basic approach

To estimate the star-discrepancy of (t, m, s)-nets we use the 'raw form' of the inequality of Erdös-Turán-Koksma for the Haar function system given in Hellekalek [6,7] (an analogous result for Walsh functions is shown in [5]). In the following, we present a concise summary of the basic approach given therein.

3.1. Step 1: Discretization with resolution $M := b^\gamma$, $\gamma \in \boldsymbol{N}$

We shall approximate $J \in \mathcal{J}$ by an inner interval $\underline{J}$ and an outer interval $\bar{J}$ in $[0, 1[^s$ where these intervals have the form $\prod_{i=1}^{s}[0, a_i/M[$, $0 \leq a_i \leq M$. This will be done in order to get a finite Haar series of the functions $f_{\underline{J}}$ and $f_{\bar{J}}$.

The approximation may be realized in the following way: Let $J := \prod_{i=1}^{s}[0, u_i[$, $\underline{J} := \prod_{i=1}^{s}[0, v_i[$ and $\bar{J} := \prod_{i=1}^{s}[0, w_i[$. If $u_i = a_i/M$, $0 \leq a_i \leq M$, we set $v_i = w_i = u_i$ otherwise let $v_i := u_i(\gamma)$ and $w_i := u_i(\gamma) + 1/M$. Therefore we get

$$|R_N(f_J, \mathcal{P})| \leq (\lambda_s(\bar{J}) - \lambda_s(\underline{J})) + \max\{|R_N(f_{\underline{J}}, \mathcal{P})|, |R_N(f_{\bar{J}}, \mathcal{P})|\}. \tag{4}$$

Using ([14], Lemma 3.9) we obtain

$$D_N^*(\mathcal{P}) \leq 1 - \left(1 - \frac{1}{M}\right)^s + \sup_{G \in \mathcal{J}_M^*} |R_N(f_G, \mathcal{P})|, \tag{5}$$

where $\mathcal{J}_M^*$ denotes the class of all subintervals G of $[0, 1[^s$ of the form $G = \prod_{i=1}^{s}[0, a_i/M[$, $0 \leq a_i \leq M$. The first term in (5) is called the *discretization error* and the second term denotes the *discrete star discrepancy*.

Remark 6.

1. Discretization. The discretization step is due to Niederreiter [11]. In this paper, he proved a variant of the Erdös-Turán-Koksma inequality for finite rational point sets (see also ([14], Section 3.2)).
2. Discrete Discrepancy. The concept of discrete discrepancy was introduced by Niederreiter as well [14]. An analogue of the result mentioned in (1) for the discrete discrepancy is given in [15], see also [7].

3.2. Step 2: Estimation of the discrete discrepancy

We consider $G := \prod_{i=1}^{s}[0, a_i/M[$. From ([7], Lemma 3.3) it follows that the Haar series of the function f_G is finite. More precisely

$$f_G(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\Delta_\gamma^*} \widehat{\mathbf{1}_G}(\boldsymbol{k}) \cdot h_{\boldsymbol{k}}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in [0,1[^s, \tag{6}$$

with Haar coefficients $\widehat{\mathbf{1}_G}(\boldsymbol{k}) := \int_{[0,1[^s} \mathbf{1}_G \cdot h_{\boldsymbol{k}}\, \mathrm{d}\lambda_s$ and $\Delta_\gamma^* := \{\boldsymbol{k} = (k_1, \dots, k_s) \in \boldsymbol{Z}^s : 0 \le k_i < M, 0 \le i \le s\} \setminus \{0\}$. Identity (6) and definition (2) yield

$$R_N(f_G, \mathcal{P}) = \sum_{\boldsymbol{k}\in\Delta_\gamma^*} \widehat{\mathbf{1}_G}(\boldsymbol{k}) \cdot R_N(h_{\boldsymbol{k}}, \mathcal{P}). \tag{7}$$

Corollary 7. *Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{N-1}$ be a point set in* $[0, 1[^s$. *From the considerations above we get the 'raw form' of the inequality of Erdös-Turán-Koksma for the Haar function system*

$$D_N^*(\mathcal{P}) \le 1 - \left(1 - \frac{1}{M}\right)^s + \sup_{G\in\mathcal{J}_M^*} \sum_{\boldsymbol{k}\in\Delta_\gamma^*} |\widehat{\mathbf{1}_G}(\boldsymbol{k})| \cdot |R_N(h_{\boldsymbol{k}}, \mathcal{P})|. \tag{8}$$

Estimates of the Haar coefficients $\widehat{\mathbf{1}_G}$ can be found in Hellekalek [6,7].

4. An application to special (t, m, s)-nets

The definition of (t, m, s)-nets $\mathcal{P}$ is closely coherent with Haar functions [17]. Hence proper estimates of the Weyl sums $R_N(h_{\boldsymbol{k}}, \mathcal{P})$ potentially yield small upper bounds of the star discrepancy. But for arbitrary (t, m, s)-nets in base b our method yields no improvement against existing estimates due to weak upper bounds of the Weyl sums in general [1].

The situation probably changes if we restrict to special construction methods called *digital nets* defined below.

Definition 8. Let $b \ge 2$ be a given base. For $1 \le i \le s$, let $\boldsymbol{C}^{(i)}$ be $m \times m$ matrices over $\boldsymbol{Z}_b$. In the following every integer n with $0 \le n < b^m$ and digit expansion $\sum_{i=0}^{m-1} n_i b^i$, $n_i \in \boldsymbol{Z}_b$, is identified with the vector $\vec{\boldsymbol{n}} = (n_0, \dots, n_{m-1})^t \in \boldsymbol{Z}_b^m$, and each $x \in [0, 1[$ with finite digit expansion $x = \sum_{i=0}^{m-1} x_i/b^{i+1}$, $x_i \in \boldsymbol{Z}_b$ is identified with $\vec{\boldsymbol{x}} = (x_0, \dots, x_{m-1})^t \in \boldsymbol{Z}_b^m$. Consider $\vec{\boldsymbol{x}}_n^{(i)} = \boldsymbol{C}^{(i)}. \vec{\boldsymbol{n}}$ for $0 \le n < b^m$ and $1 \le i \le s$. Then we obtain the following point set $\mathcal{P} \in [0, 1[^s$

$$\mathcal{P} = \{\boldsymbol{x}_n : \boldsymbol{x}_n = (x_n^{(1)}, \dots, x_n^{(s)}), \quad 0 \le n < b^m\}. \tag{9}$$

These point sets were defined in [8–10] (in a more general form). The general construction principle was introduced by Niederreiter [12–14].

Conditions ([12], Section 6) were given for $\mathcal{P}$ to be a (t, m, s)-net in base b. For example, if b is prime and $c_1^{(i)}, \dots, c_m^{(i)}$ are the row vectors of $\boldsymbol{C}^{(i)}$, then $\mathcal{P}$ is a (t, m, s)-net in base b if and only if for all $g_1, \dots, g_s \in \boldsymbol{N}_0$ with $g_1 + \cdots + g_s = m - t$, the set of vectors $\{c_j^{(i)} : 1 \le j \le g_i, 1 \le i \le s\}$ is assumed to be linearly independent over $\boldsymbol{Z}_b$. Concrete examples of digital (t, m, s)-nets can be found in [9,12].

4.1. A classical example

The conditions above yield the following simple (t, m, 2)-nets $\mathcal{P}_t$ defined by the matrices below.

$$C^{(1)} := \left|\begin{array}{l} 000\cdots000\cdots00\mathbf{1}\\ 000\cdots000\cdots0\mathbf{1}0\\ 000\cdots000\cdots\mathbf{1}00\\ \;\vdots\qquad\quad\vdots\qquad\quad\vdots\\ 000\cdots00\mathbf{1}\cdots000\\ 000\cdots0\mathbf{1}0\cdots000\\ \left.\begin{array}{l}000\cdots000\cdots000\\ \;\vdots\qquad\quad\vdots\qquad\quad\vdots\\ 000\cdots000\cdots000\end{array}\right\}t \end{array}\right|$$

$$C^{(2)} := \left|\begin{array}{l} \mathbf{1}00\cdots000\cdots000\\ 0\mathbf{1}0\cdots000\cdots000\\ 00\mathbf{1}\cdots000\cdots000\\ \;\vdots\qquad\quad\vdots\qquad\quad\vdots\\ 000\cdots\mathbf{1}00\cdots000\\ 000\cdots0\mathbf{1}0\cdots000\\ \left.\begin{array}{l}000\cdots000\cdots000\\ \;\vdots\qquad\quad\vdots\qquad\quad\vdots\\ 000\cdots000\cdots000\end{array}\right\}t \end{array}\right|$$

The net $\mathcal{P}_0 := \{\boldsymbol{x}_n = (0.n_{m-1}\dots n_0, 0.n_0\dots n_{m-1}) : 0 \le n < b^m\}$ is called the *Hammersley point set* in base b which is well-known in the theory of uniform distribution of sequences modulo one [14]. The calculation of the exact discrepancy of $\mathcal{P}_0$ is carried out in [4]. If we truncate t bits of each coordinate of the *Hammersley point set*, we obtain $\mathcal{P}_t$, $t \ge 1$. Note that for even m and $t = m/2$ the uniform lattice with 2^m points in $[0, 1[^2$ is obtained. Hence $\mathcal{P}_t$, for increasing t, $0 \le t \le m/2$, provide remarkable examples of point sets with steadily decreasing equidistribution property, from 'optimal' (0, m, 2)-nets to the classical uniform lattice.

In the following let $b = 2$, $m > 2$ and $0 \le t \le \lfloor m/2 \rfloor$. Larger values of t lead to a duplication of points. In other words, different values of n lead to the same point. The point set $\mathcal{P}_t$ matches with the Haar function system $\mathcal{H}_2$ in the sense that the Weyl sums are very small and easy to determine, see below.

4.2. Haar coefficients and Weyl Sums

The definition of (t, m, s)-nets easily yields $R_N(H_{\boldsymbol{k}}, \mathcal{P}_t) = 0$ for all $\boldsymbol{k} \in \Delta(\boldsymbol{g})$, $\boldsymbol{g} \in \boldsymbol{N}_1^2(-1, -1)$ with $\sum_{i=1}^{2}(g_i + 1) \le m - t$. The latter property and the fact that all points of $\mathcal{P}_t$ have common denominator 2^{m-t} motivate discretization (Step 1) with resolution $M := 2^\gamma$, $\gamma = m - t$.

Let $G := [0, \alpha[\times[0, \beta[$ with $\alpha = a/M$ and $\beta = b/M$, $1 \le a, b < M$. From the result above we only have to consider resolution vectors $\boldsymbol{g} = (g_1, g_2)$ with $0 \le g_1, g_2 < \gamma$ and $\gamma - 1 \le g_1 + g_2 \le 2(\gamma - 1)$. Hence for Step 2 we are concerned with the expression

$$R_N(f_G, \mathcal{P}_t) = \sum_{g_1, g_2=0,\ g_1+g_2 \ge \gamma-1}^{\gamma-1} 2^{(g_1+g_2)/2} \sum_{\boldsymbol{k}\in\Delta(\boldsymbol{g})} \widehat{\mathbf{1}_G}(\boldsymbol{k}) \cdot R_N(H_{\boldsymbol{k}}, \mathcal{P}_t). \tag{10}$$

Hellekalek ([6], Lemma 1) provides detailed information on the Haar coefficients $\widehat{\mathbf{1}_G}(\boldsymbol{k})$. The Weyl sums, for different resolutions, have been calculated in [2]. Here we need to recall only two cases

1. let $g_1 + g_2 = m - t + j$, $j \in \{-1, 0, \ldots, t-1\}$. If $j = t-1$ then $R_N(H_{\boldsymbol{k}}, \mathcal{P}_t) = 1/2^{m-1}$ for all $\boldsymbol{k} \in \Delta(\boldsymbol{g})$. For $j \leq t-2$ we obtain $R_N(H_{\boldsymbol{k}}, \mathcal{P}_t) = 0$ for all $\boldsymbol{k} \in \Delta(\boldsymbol{g})$.
2. Let $g_1 + g_2 = m + j$, $0 \leq j \leq m - 2t - 1$. Hence we obtain

$$R_N(H_{\boldsymbol{k}}, \mathcal{P}_t) = \begin{cases} 2^{-m} & : \quad k_0 = l_{j-1}, \ldots, k_{j-1} = l_0, \text{ and } x_{g_1} = y_{g_2} \\ -2^{-m} & : \quad k_0 = l_{j-1}, \ldots, k_{j-1} = l_0, \text{ and } x_{g_1} \neq y_{g_2} \\ 0 & : \quad \text{otherwise.} \end{cases} \tag{11}$$

4.3. Discrepancy estimation of $\mathcal{P}_t$

Halton and Zaremba [4] examined the ‘closed’ version of the discrepancy of $\mathcal{P}_0$ which means that they used closed intervals in Definition 1. The intervals $J_1 := [0, \alpha^*] \times [0, \beta^*]$ and $J_2 := [0, \alpha^{**}] \times [0, \beta^{**}]$ where the suprema are attained, are given by

$$\alpha^* = \frac{2^m - (-1)^m}{3(2^{m-1})} \quad \text{and} \quad \alpha^{**} = \frac{5(2^{m-2}) + (-1)^m}{3(2^{m-1})}, \tag{12}$$

where β^* and β^{**} are respectively equal to α^* and α^{**} in that order when m is even and in the reversed order when m is odd. These intervals obviously will play the central role in our considerations below.

The discrepancy estimation of $\mathcal{P}_t$ using Haar functions needs to distinguish between several cases (similar as for $\mathcal{P}_0$ in [4]). In this paper we want to give an idea, how the estimation procedure works. A detailed derivation needs much more space and therefore we want to refer to a forthcoming article.

In the following we explain the main steps to achieve the estimate. Therefore, we will limit to the case of even m only.

Step 1: Since we know the ‘maximal’ intervals for the discrepancy of $\mathcal{P}_0$, we can derive the corresponding intervals for the *discrete discrepancy* of $\mathcal{P}_t$. We slightly have to change the discretization step for $\mathcal{P}_0$ and apply the inner interval $\underline{J} := [0, \alpha^*[\times[0, \alpha^*[$ and outer interval $\bar{J} := [0, \alpha'[\times[0, \beta'[$ with $\alpha' = \beta' = \alpha^* + 1/2^m = 2(2^m + 1)/3(2^m)$. It turns out that the discrete discrepancy for $\mathcal{P}_0$ equals $|R_N(f_{\bar{J}}, \mathcal{P}_0)|$.

Step 2: The main effort to obtain our estimate lies in the derivation of the discrete discrepancy. Therefore we need to constitute the binary expansion of the marginal values of the inner and outer intervals, since the exact calculation of the Haar coefficients from Hellekalek [6] depends on the digits of these numbers. The determination of the Haar coefficients specifies the relevant vectors $\boldsymbol{k}$ in the right sum of (10). With these vectors $\boldsymbol{k}$ we are able to derive the corresponding Weyl sums for the different cases (1) and (2).

The magnitude of the sum in (10) depends only on the Haar coefficients the Weyl sums are constants. Hence we can derive the ‘maximal’ intervals for the discrete discrepancy of $\mathcal{P}_t$. Changing m to $m - 2t$ in $\bar{J}$ above yields $\bar{J} := [0, \alpha[\times[0, \beta[$, $\alpha = \phi/2^t + \delta$, $\beta = \phi'/2^t + \delta$, $0 \leq \phi, \phi' < 2^t$, $\delta = 2(2^{m-t} + 1)/3(2^{m-t})$.

A careful evaluation of (10) yields the following estimate for even m:

$$D_N^*(\mathcal{P}_t) \leq 1 - \left(1 - \frac{1}{2^{m-t}}\right)^2 + \frac{1}{3}\frac{m-2t}{2^m} + \frac{1}{9}\frac{1}{2^m} - \frac{1}{9}\frac{1}{2^{2(m-t)}} \tag{13}$$

Remark 9.

1. Let m be even and $t = 0$. Since we know the interval where the maximum in (4) occurs we can calculate the exact discretization error $(\lambda_s(\bar{J}) - \lambda_s(\underline{J})) = (2^{m+2} - 1)/3(2^{2m})$. The therefore we get

$$D_N^*(\mathcal{P}_0) \geq \frac{1}{3}\frac{m}{2^m} + \frac{13}{9}\frac{1}{2^m} - \frac{4}{9}\frac{1}{2^{2m}} \tag{14}$$

From [4] we know that the latter expression equals the exact discrepancy of $\mathcal{P}_0$, hence our estimate is best possible. Further, from ([14], Theorem. 3.14) we deduce that $D_N^*(\mathcal{P}_{m/2}) = 1 - (1 - 1/2^{m/2})^2$.
2. Consider the point set $\mathcal{P} = \mathcal{P}_0$. There are two different intervals where the supremum in (1) occurred. In our case we obviously get the same number of 'maximal' intervals defined by α' and $a'' = 1 - \alpha'$. For $\mathcal{P} = \mathcal{P}_t$, $0 \leq t \leq m/2$ there are 2^{2t+1} such intervals.

The graphics in Fig. 1 indicate property (2) and, in an impressive way, exhibit the structural behavior of local discrepancy for our nets. Each rectangular block represents a marginal point of an interval J. The level of each block exhibits the magnitude of the local discrepancy $|R_N(f_J, \mathcal{P})|$. Fig. 1 considers the cases $m = 6$, $t = 1$ (upper graphics) and $t = 2$ (lower graphics).

4.4. Summary

An application of Haar function systems provides interesting methods to estimate the (star) discrepancy of certain (t, m, s)-nets. In the general case, i.e. for arbitrary nets, we obtained no improvements against existing results. But for special digital nets it is possible to derive excellent (sometimes best possible) bounds for the star-discrepancy. Our method also leads to the evaluation of the exact calculation of discrete discrepancy, a natural measure of equidistribution for finite precision point sets such as digital nets. For our digital nets $\mathcal{P}_t$ we finally get (if we also consider odd numbers m)

Proposition 10. *For $\mathcal{P}_t$, $0 \leq t \leq \lfloor m/2 \rfloor$, the digital $(t, m, 2)$-nets in base $b = 2$, defined in Section 4.1 we get*

$$D_N^*(\mathcal{P}_t) \leq 1 - \left(1 - \frac{1}{2^{m-t}}\right)^2 + \frac{1}{3}\frac{m-2t}{2^m} + \frac{1}{9}\frac{1}{2^m} - \frac{(-1)^m}{9}\frac{1}{2^{2(m-t)}}$$

Hence, as expected, with increasing parameter $0 \leq t \leq \lfloor m/2 \rfloor$ the distribution quality of the nets $\mathcal{P}_t$ steadily decreases.

Remark 11. If we change the 'quality criterion', i.e. we use, instead of indicator functions, a different function g in $R_N(g, \mathcal{P}_t)$, then we get the latter property in reversed order (the distribution with respect to $R_N(g, \mathcal{P}_t)$ increases). Let $m > 2$ be even. Consider the rapidly converging Haar series $g(\boldsymbol{x}) := \sum_{g_1, g_2 \geq 0} \sum_{\boldsymbol{k} \in \Delta(\boldsymbol{g})} r(\boldsymbol{k})^{3/2} h_{\boldsymbol{k}}$ where $r(k) = 0$, if there exists a $g_i \in \{m/2, \ldots, m-1\}$, and $r(\boldsymbol{k}) = 2^{(g_1+g_2)}$ otherwise. From ([1], Theorem 3.3.3) we get $R_N(g, \mathcal{P}_t) = 4/(2^m + 2^{2m})$ for $t = m/2$ and $R_N(g, \mathcal{P}_s) = 2(m-s)/2^m + 4/(2^m + 2^{2m})$ for $0 \leq s < t$, and therefore $R_N(g, \mathcal{P}_0) > R_N(g, \mathcal{P}_1) > \cdots > R_N(g, \mathcal{P}_t)$.

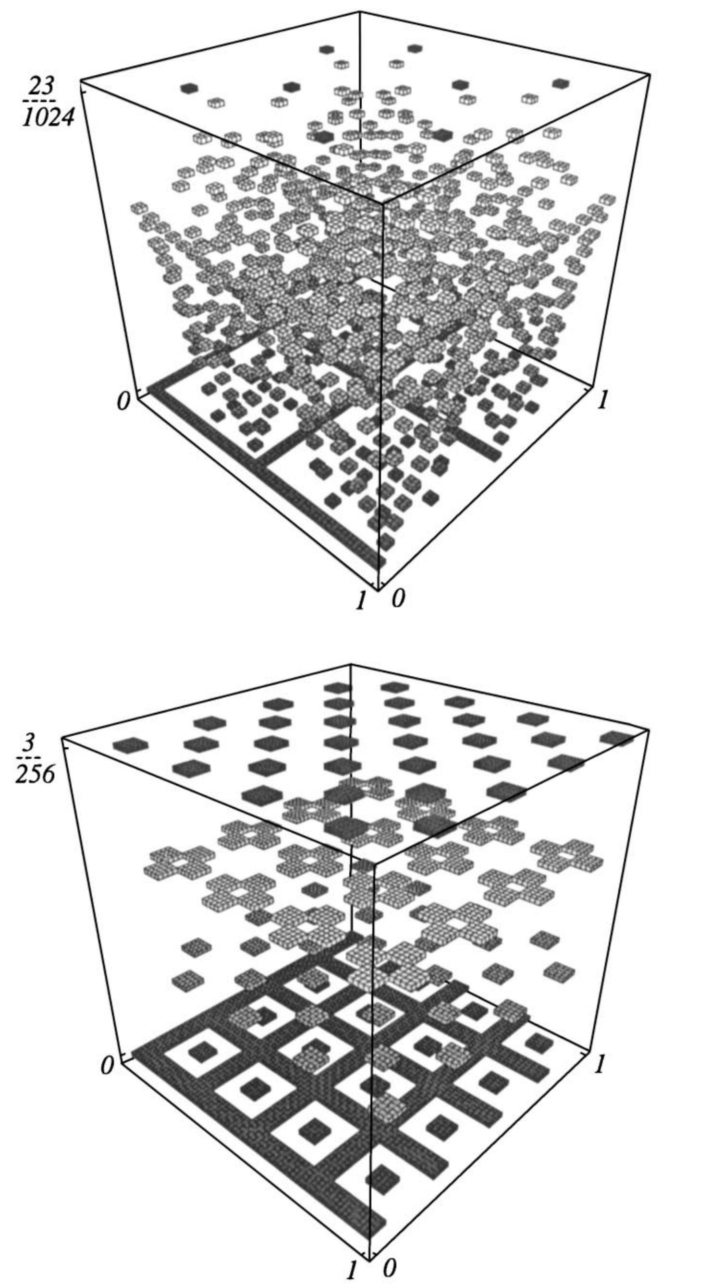

Fig. 1. Behavior of the local discrepancy $|R_N(f_J, \mathcal{P})|$ for $m = 6$ and $t = 1, 2$.

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